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Abstract. We prove a bound for quintilinear sums of Kloosterman sums, with congruence conditions on the "smooth" summation variables. This generalizes classical work of Deshouillers and Iwaniec, and is key to obtaining power-saving error terms in applications, notably the dispersion method.

As a consequence, assuming the Riemann hypothesis for Dirichlet L-functions, we prove power-saving error term in the Titchmarsh divisor problem of estimating $\sum_{p\leq x}\tau(p-1)$. Unconditionally, we isolate the possible contribution of Siegel zeroes, showing it is always negative. Extending work of Fouvry and Tenenbaum, we obtain power-saving in the asymptotic formula for $\sum_{n\leq x}\tau_k(n)\tau(n+1)$, reproving a result announced by Bykovskiĭ and Vinogradov by a different method. The gain in the exponent is shown to be independent of k if a generalized Lindelöf hypothesis is assumed.

1. Introduction

Understanding the joint multiplicative structure of pairs of neighboring integers such as $(n,n+1)$ is an outstanding problem in multiplicative number theory. A quantitative way to look at this question is to try to estimate sums of the type

$$\sum_{n\leq x} f(n)g(n+1) \tag{1.1}$$

when $f,g:\mathbf{N}\to\mathbf{C}$ are two functions that are of multiplicative nature – multiplicative functions for instance, or the characteristic function of primes. In this paper we are motivated by two instances of the question (1.1): the Titchmarsh divisor problem, and correlation of divisor functions.

In what follows, $\tau(n)$ denotes the number of divisors of the integer n, and more generally, $\tau_k(n)$ denotes the number of ways one can write n as a product of k positive integers. Studying the function τ_k gives some insight into the factorisation of numbers[1], which is deeper but more difficult to obtain as k grows.

1.1. **The Titchmarsh divisor problem.** One would like to be able to evaluate, for $k\geq 2$, the sum

$$\sum_{p\leq x}\tau_k(p-1) \tag{1.2}$$

where p denotes primes. *A priori*, this would require understanding primes up to x in arithmetic progressions of moduli up to $x^{1-1/k}$. The case $k\geq 3$ seems far from reach of current methods, so we consider $k=2$.

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[1]There are a number of formulas relating the characteristic function of primes to linear combination of divisor-like functions, for instance Heath-Brown's identity [HB82].

In place of (1.2), one may consider

$$T(x) := \sum_{1<n\leq x} \Lambda(n)\tau(n-1)$$

where Λ is the von Mangoldt function [IK04, formula (1.39)]. In 1930, Titchmarsh [Tit30] first considered the problem, and proved $T(x) \sim C_1 x \log x$ for some constant $C_1 > 1$ under the assumption that the Riemann hypothesis holds for all Dirichlet L-functions. This asymptotics was proved unconditionally by Linnik [Lin63] using his so-called dispersion method. Simpler proofs were later given by Rodriquez [Rod65] and Halberstam [Hal67] using the theorems of Bombieri-Vinogradov and Brun-Titchmarsh. Finally the most precise known estimate was proved independently by Bombieri–Friedlander–Iwaniec [BFI86] and Fouvry [Fou85]. To state their result, let us denote

$$C_1 := \prod_p \Big(1 + \frac{1}{p(p-1)}\Big), \qquad C_2 := \sum_p \frac{\log p}{1+p(p-1)}.$$

Theorem A (Fouvry [Fou85], Bombieri–Friedlander–Iwaniec [BFI86])**.** *For all $A > 0$ and all $x \geq 3$,*

$$T(x) = C_1 x\Big\{\log x + 2\gamma - 1 - 2C_2\Big\} + O_A\Big(x/(\log x)^A\Big).$$

In this statement, γ denotes Euler's contant. See also [Fel12, Fio12a] for generalizations in arithmetic progressions; and [ABSR15] for an analog in function fields.

The error term in Theorem A is due to an application of the Siegel-Walfisz theorem [IK04, Corollary 5.29]. One could wonder whether assuming the Riemann Hypothesis generalized to Dirichlet L-functions (GRH) allows for power-saving error term to be obtained (as is the case for the prime number theorem in arithmetic progressions [MV07, Corollary 13.8]). The purpose of this paper is to prove that such is indeed the case.

Theorem 1.1. *Assume GRH. Then for some $\delta > 0$ and all $x \geq 2$,*

$$T(x) = C_1 x\Big\{\log x + 2\gamma - 1 - 2C_2\Big\} + O(x^{1-\delta}).$$

Unconditionally, we quantify the influence of hypothetical Siegel zeroes. Define, for $q \geq 1$,

$$C_1(q) := \frac{1}{\varphi(q)} \prod_{p\nmid q} \Big(1 + \frac{1}{p(p-1)}\Big), \qquad C_2(q) := \sum_{p\nmid q} \frac{\log p}{1+p(p-1)}$$

where φ is Euler's totient function. Note that $C_1 = C_1(1)$ and $C_2 = C_2(1)$.

Theorem 1.2. *There exist $b > 0$ and $\delta > 0$ such that*

$$\begin{aligned} T(x) = {} & C_1 x\Big\{\log x + 2\gamma - 1 - 2C_2\Big\} \\ & - C_1(q)\frac{x^\beta}{\beta}\Big\{\log\Big(\frac{x}{q^2}\Big) + 2\gamma - \frac{1}{\beta} - 2C_2(q)\Big\} + O\Big(x e^{-\delta\sqrt{\log x}}\Big). \end{aligned}$$

The second term is only to be taken into account if there is a primitive character χ (mod q) with $q \leq e^{\sqrt{\log x}}$ whose Dirichlet L-function has a real zero β with $\beta \geq 1 - b/\sqrt{\log x}$.

By partial summation, one deduces

Corollary 1.3. *In the same notation as Theorem 1.2,*
$$\sum_{p\le x}\tau(p-1)=C_1\{x+2\,\mathrm{li}(x)(\gamma-C_2)\}-C_1(q)\{\frac{x^\beta}{\beta}+2\,\mathrm{li}(x^\beta)(\gamma-\log q-C_2(q))\}+O(x\mathrm{e}^{-\delta\sqrt{\log x}}).$$

The method readily allows for more general shifts $\tau(p-a)$, $0<|a|\le x^\delta$ (*cf.* [Fio12b, Corollary 3.4] for results on the uniformity in a). In the case $a=1$, or more generally when a is a perfect square, we have an unconditional inequality.

Corollary 1.4. *With an effective implicit constant, we have*
$$\sum_{p\le x}\tau(p-1)\le C_1\{x+2\,\mathrm{li}(x)(\gamma-C_2)\}+O(x\mathrm{e}^{-\delta\sqrt{\log x}}).$$

We conclude our discussion of the Titchmarsh divisor problem by mentioning the important work of Pitt [Pit13], who proves $\sum_{p\le x}a(p-1)\ll x^{1-\delta}$ for the sequence $(a(n))$ of Fourier coefficients of an integral weight holomorphic cusp form (which is a special case of (1.1) when the $(a(n))$ are Hecke eigenvalues). It is a striking feature that power-saving can be proved *unconditionally* in this situation.

1.2. **Correlation of divisor functions.** Another instance of the problem (1.1) is the estimation, for integers $k,\ell\ge 2$, of the quantity
$$\mathcal{T}_{k,\ell}(x):=\sum_{n\le x}\tau_k(n)\tau_\ell(n+1).$$
The conjectured estimate is of the shape
$$\mathcal{T}_{k,\ell}(x)\sim C_{k,\ell}x(\log x)^{k+\ell-2}$$
for some constants $C_{k,\ell}>0$. The case $k=\ell$ is of particular interest when one looks at the $2k$-th moment of the Riemann ζ function [Tit86, §7.21] (see also [CG01]): in that context, the size of the error term is a non-trivial issue, as well as the uniformity with which one can replace $n+1$ above by $n+a$, $a\ne 0$. Current methods are ineffective when $k,\ell\ge 3$, so we focus on the case $\ell=2$. Let us denote
$$\mathcal{T}_k(x):=\sum_{n\le x}\tau_k(n)\tau(n+1).$$
There has been several works on the estimation of $\mathcal{T}_k(x)$. There are nice expositions of the history of the problem in the papers of Heath-Brown [HB86] and Fouvry-Tenenbaum [FT85]. The latest published results may be summarized as follows.

Theorem B. *There holds:*
$$\mathcal{T}_2(x)=xP_2(\log x)+O_\varepsilon(x^{2/3+\varepsilon}),\qquad \text{([DI82a])},$$
$$\mathcal{T}_3(x)=xP_3(\log x)+O(x^{1-\delta}),\qquad \text{([Des82], [Top15])},$$
$$\mathcal{T}_k(x)=xP_k(\log x)+O_k(x\mathrm{e}^{-\delta\sqrt{\log x}})\quad \textit{for fixed } k\ge 4,\qquad \text{([FT85])}.\tag{1.3}$$
Here $\varepsilon>0$ is arbitrary, $\delta>0$ is some constant depending on k, and P_k is an explicit degree k polynomial.

The error term of (1.3) resembles that in the distribution of primes in arithmetic progressions, where it is linked to the outstanding problem of zero-free regions of L-functions. However there is no such process at work in (1.3), leaving one to wonder if power-saving can be achieved. In [BV87], Bykovskiĭ and Vinogradov announce results implying
$$\mathcal{T}_k(x)=xP_k(\log x)+O_k(x^{1-\delta/k})\qquad (k\ge 4, x\ge 2)\tag{1.4}$$

for some absolute $\delta > 0$, and sketch ideas of a proof. The proposed argument, in a way, is dual to the method adopted in [FT85][2] (which is related to earlier work of Motohashi [Mot76]). Here we take up the method of [FT85] and prove an error term of the same shape.

Theorem 1.5. *For some absolute $\delta > 0$, the estimate* (1.4) *holds.*

In view of [BV87], Theorem 1.5 is not new. However the method is somewhat different. In the course of our arguments, the analytic obstacle to obtaining an error term $O_k(x^{1-\delta})$ (δ independent of k) in the estimate (1.4) will appear clearly: it lies in the estimation of sums of the shape $\sum_{n\leq x} \tau_k(n)\chi(n)$ for Dirichlet characters χ of small conductors. This issue is know to be closely related to the growth of Dirichlet L-functions inside the critical strip [FI05].

Theorem 1.6. *Assume that Dirichlet L-functions satisfy the Lindelöf hypothesis, meaning $L(\frac12 + it, \chi) \ll_\varepsilon (q(|t|+1))^\varepsilon$ for $t \in \mathbf{R}$ and $\chi \pmod q$. Then for some* absolute $\delta > 0$,

$$\mathcal{T}_k(x) = P_k(\log x) + O_k(x^{1-\delta}) \qquad (k \geq 4,\ x \geq 2) \tag{1.5}$$

The standard conjecture for the error term in the previous formula is $O_{k,\varepsilon}(x^{1/2+\varepsilon})$. We have not sought optimal values for δ in Theorems 1.5 and 1.6. In the case of (1.4), the method of [BV87] seems to yield much better numerical results.

Our method readily allows to replace the shift $n+1$ in Theorem 1.5 by $n+a$, $0 < |a| \leq x^\delta$ with an exponent *independent of k*. We give some explanations in Section 7.3 below regarding this point.

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2. Overview

The method at work in Theorems 1.1, 1.2 and 1.5 is the dispersion method, which was pioneered by Linnik [Lin63] and studied intensively in groundbreaking work of Bombieri, Fouvry, Friedlander and Iwaniec [Fou82, FI83, BFI86] on primes in arithmetic progressions. It has received a large publicity recently with the breakthrough of Zhang [Zha14] (see also [PCF$^+$14]), giving the first proof of the existence of infinitely many bounded gaps between primes (which was shown later by Maynard [May15] and Tao (unpublished) not to require such strong results).

In our case, by writing $\tau(n)$ as a convolution of the constant function 1 with itself, the problem is reduced to estimating the mean value of $\Lambda(n)$ or $\tau_k(n)$ when $n \leq x$ runs over arithmetic progressions $\pmod q$, with an average over q. It is crucial that the uniformity be good enough to average over $q \leq \sqrt{x}$. In the case of $\Lambda(n)$, that is beyond what can currently be done for individual moduli q, even assuming the GRH. The celebrated theorem of Bombieri-Vinogradov [IK04, Theorem 17.1] allows to exploit

[2]In [Mot76, FT85], the authors study the distribution of $\tau_k(n)$ in progressions of moduli up to $x^{1/2}$, while in [BV87] the authors address the distribution of $\tau(n)$ in progressions of moduli up to $x^{1-1/k}$.

the averaging over q, but if one wants error terms at least as good as $O(x/(\log x)^2)$ for instance, it barely fails to be useful.

Linnik's dispersion method [Lin63], which corresponds at a technical level to an acute use of the Cauchy–Schwarz inequality, offers the possibility for such results, on the condition that one has good bounds on some types of exponential sums related to Kloosterman sums. One then appeals to Weil's bound [Wei48], or to the more specific but stronger bounds of Deshouillers-Iwaniec [DI82b] which originate from the theory of modular forms through Kuznetsov's formula.

The Deshouillers-Iwaniec bounds apply to exponential sums of the following kind:

$$\sum_{\substack{c,d,n,r,s \\ (rd,sc)=1}} b_{n,r,s} g(c,d) \mathrm{e}\Big(n\frac{\overline{rd}}{sc}\Big)$$

where c, d, n, r, s are integers in specific intervals, $(b_{n,r,s})$ is a generic sequence, and $g(c,d)$ depends in a smooth way on c and d. Here and in what follows, $\mathrm{e}(x)$ stand for $\mathrm{e}^{2\pi i x}$, and $\overline{rd}$ denotes the multiplicative inverse of $rd \pmod{sc}$ (since $\mathrm{e}(x)$ is of period 1, the above is well-defined). It is crucial that the variables c and d are attached to a smooth weight $g(c,d)$: for the variable d, in order to reduce to complete Kloosterman sums $\pmod{sc}$; and for the variable c, because the object that arises naturally in the context of modular forms is the average of Kloosterman sums over moduli (with smooth weight).

In the dispersion method, dealing with largest common divisors (appearing through the Cauchy–Schwarz inequality) causes some issues. The most important of these is that the phase function that arises in the course of the argument takes a form similar to

$$\mathrm{e}\Big(n\frac{\overline{rd}}{sc} + \frac{\overline{cd}}{q}\Big) \tag{2.1}$$

rather than the above. Here q can be considered small and fixed, but even then, the second term oscillates chaotically.

Previous works avoided the issue altogether by using a sieve beforehand in order to reduce to the favourable case $q = 1$. Two error terms are then produced, which take the form

$$\mathrm{e}^{-\delta(\log x)/\log z} + z^{-1}$$

where $z \leq x$ is a parameter. Roughly speaking, the first term corresponds to sieving out prime factors smaller than z, with the consequence that the "bad" variable q above is either 1 or larger than z. The second term corresponds to a trivial bound on the contribution of $q > z$. The best error term one can achieve in this way is $\mathrm{e}^{-\delta\sqrt{\log x}}$, whence the estimate (1.3).

By contrast, in the present paper, we transpose the work of Deshouillers-Iwaniec in a slightly more general context, which allows to encode phases of the kind (2.1). More specifically, whereas Deshouillers and Iwaniec worked with modular forms with trivial multiplier system, we find that working with multiplier systems defined by Dirichlet characters allows one to encode congruence conditions $\pmod q$ on the "smooth" variables c and d. This is partly inspired by recent work of Blomer and Milićević [BM15a]. The main result, which extends [DI82b, Theorem 12] and has potential for applications beyond the scope of the present paper, is the following.

Theorem 2.1. *Let $C, D, N, R, S \geq 1$, and $q, c_0, d_0 \in \mathbf{N}$ be given with $(c_0 d_0, q) = 1$. Let $(b_{n,r,s})$ be a sequence supported inside $(R, 2R] \times (S, 2S] \times (0, N] \cap \mathbf{N}^3$. Let $g : \mathbf{R}_+^5 \to$*

$\mathbf{C}$ *be a smooth function compactly supported in* $]C,2C]\times]D,2C]\times(\mathbf{R}_+^*)^3$, *satisfying the bound*

$$\frac{\partial^{\nu_1+\nu_2+\nu_3+\nu_4+\nu_5}g}{\partial c^{\nu_1}\partial d^{\nu_2}\partial n^{\nu_3}\partial r^{\nu_4}\partial s^{\nu_5}}(c,d,n,r,s)\ll_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5}\{c^{-\nu_1}d^{-\nu_2}n^{-\nu_3}r^{-\nu_4}s^{-\nu_5}\}^{1-\varepsilon_0} \tag{2.2}$$

for some small $\varepsilon_0>0$ *and all fixed* $\nu_j\geq 0$. *Then*

$$\begin{aligned}\sum_{\substack{c\\ c\equiv c_0 \text{ and } d\equiv d_0 \pmod q\\ (qrd,sc)=1}}\sum_d\sum_n\sum_r\sum_s & b_{n,r,s}g(c,d,n,r,s)\mathrm{e}\Big(n\frac{\overline{rd}}{sc}\Big)\\ &\ll_{\varepsilon,\varepsilon_0}(CDNRS)^{\varepsilon+O(\varepsilon_0)}qK(C,D,N,R,S)\|b_{N,R,S}\|_2,\end{aligned} \tag{2.3}$$

where $\|b_{N,R,S}\|_2=\Big(\sum_{n,r,s}|b_{n,r,s}|^2\Big)^{1/2}$ *and*

$$K(C,D,N,R,S)^2=CS(RS+N)(C+RD)+C^2DS\sqrt{(RS+N)R}+D^2NRS^{-1}.$$

We have made no attempt to optimize the dependence in q. In all of the applications considered here, we only apply the estimate (2.3) for small values of q, say $q=O((CDNRS)^{\varepsilon_1})$ for some small $\varepsilon_1>0$. Such being the case, the reader might still wonder why the bound tends to grow with q. The main reason is that upon completing the sum over d, we obtain a Kloosterman sum to modulus scq, which grows with q.

In the footsteps of previous work [Dra15], for the proof of our equidistribution results, we separate from the outset of the argument the contribution of characters of small conductors (which is typically well-handled by complex-analytic methods). We only apply the dispersion method to the contribution of characters of large conductors. There is considerable simplification coming from the fact that no "Siegel-Walfisz"-type hypothesis is involved in the latter, which allows us to focus on the combinatorial aspect of the method[3].

In Section 3, we state a few useful lemmas. In Section 4, we adapt the arguments of [DI82b] to prove Theorem 2.1. In Section 5, we employ a variant of the dispersion method to obtain equidistribution for binary convolutions in arithmetic progressions. In Sections 6 and 7, we derive Theorems 1.1, 1.2, 1.5 and 1.6.

Notations. We use the convention that the letter ε denotes a positive number that can be chosen arbitrarily small and whose value may change at each occurence. The letter $\delta>0$ will denote a positive number whose value may change from line to line, and whose dependence on various parameters will be made clear by the context.

The Fourier transform $\widehat{f}$ of a function f is by definition

$$\widehat{f}(\xi)=\int_{\mathbf{R}}f(t)\mathrm{e}(\xi t)\mathrm{d}t.$$

If f is smooth and compactly supported, the above is well-defined and there holds

$$f(t)=\int_{\mathbf{R}}\widehat{f}(\xi)\mathrm{e}(-\xi t)\mathrm{d}\xi.$$

[3]It is more straightforward to study the mean value of $\tau_k(n)$ in arithmetic progressions of small moduli, than a k-fold convolution of slowly oscillating sequences, each supported on a dyadic interval.

If moreover f is supported inside $[-M,M]$ for some $M \geq 1$ and $\|f^{(j)}\|_\infty \ll M^{-j}$ for $j \in \{0,2\}$, then we have

$$\hat{f}(\xi) \ll \frac{M}{1+(M\xi)^2}.$$

3. Lemmas

In this section we group a few useful lemmas. The first is the Poisson summation formula, which is very effective at estimating the mean value of a smooth function along arithmetic progressions.

Lemma 3.1 ([BFI86, Lemma 2])**.** *Let $M \geq 1$ and $f : \mathbf{R} \to \mathbf{C}$ be a smooth function supported on an interval $[-M,M]$ satisfying $\|f^{(j)}\|_\infty \ll_j M^{-j}$ for all $j \geq 0$. For all $q \geq 1$ and $(a,q)=1$, with $H := q^{1+\varepsilon}/M$, we have*

$$\sum_{m \equiv a \pmod q} f(m) = \frac{1}{q}\sum_{|h|\leq H} \hat{f}\Big(\frac{h}{q}\Big)\mathrm{e}\Big(\frac{-ah}{q}\Big) + O_\varepsilon\Big(\frac{1}{q}\Big).$$

The next lemma is a very useful theorem of Shiu [Shi80, Theorem 2] and gives an upper bound of the right order of magnitude for sums of $\tau_k(n)$ in short intervals and arithmetic progressions of large moduli. It is an analog of the celebrated Brun-Titchmarsh inequality [IK04, Theorem 6.6]. We quote a special case.

Lemma 3.2 ([Shi80, Theorem 2])**.** *For $k \geq 2$, $x \geq 2$, $x^{1/2} \leq y \leq x$, $(q,a) \in \mathbf{N}$ with $(a,q)=1$ and $q \leq x^{3/4}$,*

$$\sum_{\substack{x-y<n\leq x\\ n\equiv a \pmod q}} \tau_k(n) \ll_k \frac{y}{q}\Big(\frac{\varphi(q)}{q}\log x\Big)^{k-1}.$$

The next lemma is the classical form of the multiplicative large sieve inequality [IK04, Theoreme 7.13].

Lemma 3.3. *Let (a_n) be a sequence of numbers, and $N, M, Q \geq 1$. Then*

$$\sum_{q\leq Q}\frac{q}{\varphi(q)}\sum_{\substack{\chi \pmod q\\ \chi \text{ primitive}}}\Big|\sum_{M<n\leq M+N} a_n\chi(n)\Big| \leq (Q^2+N-1)\sum_{N<n\leq N+M}|a_n|^2.$$

We quote from [Har11, Number Theory Result 1] the following version of the Pólya-Vinogradov inequality with an explicit dependence on the conductor.

Lemma 3.4. *Let $\chi \pmod q$ be a character of conductor $1 \neq r|q$, and $M, N \geq 1$. Then*

$$\sum_{M<n\leq M+N}\chi(n) \ll \tau(q/r)\sqrt{r}\log r.$$

4. Sums of Kloosterman sums in arithmetic progressions

Theorem 2.1 is proved by a systematic use of the Kuznetsov formula, which establishes a link between sums of Kloosterman sums and Fourier coefficients of holomorphic and Maaß cusp forms. There is numerous bibliography about this theory; we refer the reader to the books [Iwa02, Iwa95] and to chapters 14–16 of [IK04] for references.

Most of the arguments in [DI82b] generalizes without the need for substantial new ideas. We will introduce the main notations, and of course provide the required new arguments; but we will refer to [DI82b] for the parts of the proofs that can be transposed *verbatim*.

4.1. **Setting.**

4.1.1. *Kloosterman sums.* Let $q \geq 1$. The setting is the congruence subgroup

$$\Gamma = \Gamma_0(q) := \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}), c \equiv 0 \pmod{q} \Big\}.$$

Let χ be a character modulo $q_0 | q$, and $\kappa \in \{0, 1\}$ such that $\chi(-1) = (-1)^\kappa$. We warn the reader that the variable q has a different meaning in Sections 4.1 and 4.2, than in the statement of Theorem 2.1 (where it corresponds to qrs). The character χ induces a multiplier (*i.e.* here, a multiplicative function) on Γ by

$$\chi\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big) = \chi(d).$$

The *cusps* of Γ are Γ-equivalence classes of elements $\mathbf{R} \cup \{\infty\}$ that are parabolic, *i.e.* each of them is the unique fixed point of some element of Γ. They correspond to cusps on a fundamental domain. A set of representatives is given by rational numbers u/w where $1 \leq w$, $w|q$, $(u, w) = 1$ and u is determined $\pmod{(w, q/w)}$.

For each cusp $\mathfrak{a}$, let $\Gamma_\mathfrak{a}$ denote the stabilizer of $\mathfrak{a}$ for the action of Γ. A *scaling matrix* is an element $\sigma_\mathfrak{a} \in SL_2(\mathbf{R})$ such that $\sigma_\mathfrak{a}\infty = \mathfrak{a}$ and

$$\Big\{ \sigma_\mathfrak{a} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \sigma_\mathfrak{a}^{-1}, b \in \mathbf{Z} \Big\} = \Gamma_\mathfrak{a}.$$

Whenever $\mathfrak{a} = u/w$ with $u \neq 0$, $(u, w) = 1$ and $w|q$, one can choose

$$\sigma_\mathfrak{a} = \begin{pmatrix} \mathfrak{a}\sqrt{[q, w^2]} & 0 \\ \sqrt{[q, w^2]} & (\mathfrak{a}\sqrt{[q, w^2]})^{-1} \end{pmatrix}. \tag{4.1}$$

A cusp $\mathfrak{a}$ is said to be *singular* if $\chi(\gamma) = 1$ for any $\gamma \in \Gamma_\mathfrak{a}$. When $\mathfrak{a} = u/w$ with u and w as above, then this merely means that χ has conductor dividing $q/(w, q/w)$. The point at infinity is always a singular cusp, with stabilizer

$$\Gamma_\infty = \Big\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \Big\}.$$

For any pair of singular cusps $\mathfrak{a}, \mathfrak{b}$ and any associated scaling matrices $\sigma_\mathfrak{a}, \sigma_\mathfrak{b}$, define the set of moduli

$$\mathcal{C}(\mathfrak{a}, \mathfrak{b}) := \Big\{ c \in \mathbf{R}_+^* : \ \exists a, b, d \in \mathbf{R}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{b} \Big\}.$$

This set actually only depends on $\mathfrak{a}$ and $\mathfrak{b}$. For all $c \in \mathcal{C}(\mathfrak{a}, \mathfrak{b})$, let $\mathcal{D}_{\mathfrak{a}\mathfrak{b}}(c)$ be the set of real numbers d with $0 < d \leq c$, such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{b}$$

for some $a, b \in \mathbf{R}$. For each such d, a is uniquely determined $\pmod{c}$.

For any integers $m, n \geq 0$, and any $c \in \mathcal{C}(\mathfrak{a}, \mathfrak{b})$, the Kloosterman sum is defined as

$$S_{\sigma_\mathfrak{a}\sigma_\mathfrak{b}}(m, n; c) = \sum_{d \in \mathcal{D}_{\mathfrak{a}\mathfrak{b}}(c)} \overline{\chi}(\sigma_\mathfrak{a} \left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right) \sigma_\mathfrak{b}^{-1}) \mathrm{e}\Big(\frac{am + dn}{c} \Big)$$

where $\left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right)$ denotes any matrix γ having lower row (c, d) such that $\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1} \in \Gamma$. This is well-defined by our hypotheses that $\mathfrak{a}$ and $\mathfrak{b}$ are singular. This definition allows for a great deal of generality. We quote from [DI82b, section 2.1] the remark that the Kloosterman sums essentially depend only on the cusps $\mathfrak{a}$, $\mathfrak{b}$, and only mildly on the scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$, in the following sense. If $\widetilde{\mathfrak{a}}$ and $\widetilde{\mathfrak{b}}$ are two cusps respectively Γ-equivalent to $\mathfrak{a}$ and $\mathfrak{b}$, with respective scaling matrices $\widetilde{\sigma_{\mathfrak{a}}}$ and $\widetilde{\sigma_{\mathfrak{b}}}$, then there exist real numbers t_1 and t_2, independent of m or n, such that

$$S_{\sigma_{\mathfrak{a}}\sigma_{\mathfrak{b}}}(m,n;c) = \mathrm{e}(mt_1+nt_2) S_{\widetilde{\sigma_{\mathfrak{a}}}\widetilde{\sigma_{\mathfrak{b}}}}(m,n;c).$$

Moreover, the converse fact holds, that for any reals t_1, t_2, any cusps $\mathfrak{a}$ and $\mathfrak{b}$, and any scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$, there exist scaling matrices $\widetilde{\sigma_{\mathfrak{a}}}$ and $\widetilde{\sigma_{\mathfrak{b}}}$ associated to $\mathfrak{a}$ and $\mathfrak{b}$ such that the equality above holds. This rather simple fact is of tremendous help because all of the results obtained through the Kuznetsov formula are uniform with respect to the scaling matrices, so that one can encode oscillating factors depending of m and n at no cost (it is crucial for separation of variables). Whenever the context is clear enough, we write

$$S_{\mathfrak{a}\mathfrak{b}}(m,n;c)$$

without reference to the scaling matrices.

The first example is $\mathfrak{a} = \mathfrak{b} = \infty$ and $\sigma_{\mathfrak{a}} = \sigma_{\mathfrak{b}} = 1$. Then $\mathcal{C}(\infty,\infty) = q\mathbf{N}$ and

$$S_{\infty\infty}(m,n;c) = \sum_{d \pmod{c}^\times} \overline{\chi}(d) \mathrm{e}\Big(\frac{\overline{d}m+dn}{c}\Big) \qquad (c \in q\mathbf{N}) \tag{4.2}$$

is the usual (twisted) Kloosterman sum. Here and in the rest of the paper, we write $\pmod{c}^\times$ to mean a primitive residue class $\pmod{c}$.

The next example that we need is the case $\mathfrak{a} = \mathfrak{b}$. The following is an extension of [DI82b, Lemma 2.5]. It is proven in an identical way, so we omit the details.

Lemma 4.1. *Assume $\mathfrak{a} = u/w$ is a cusp with $(u,w) = 1$, $w|q$ and $u \neq 0$. Assume that $\mathfrak{a}$ is singular. Choose the scaling matrix as in* (4.1)*. Then $\mathcal{C}(\mathfrak{a},\mathfrak{a}) = \frac{q}{(w,q/w)}\mathbf{N}$, and if $c = \gamma q/(w,q/w)$ for some $\gamma \in \mathbf{N}$,*

$$S_{\mathfrak{a}\mathfrak{a}}(m,n;c) = \mathrm{e}\Big((w,q/w)\frac{m-n}{uq}\Big) \sideset{}{^*}\sum_{\delta \pmod{c}} \overline{\chi}\Big(\alpha + u\frac{\alpha\delta-1}{\gamma}\Big) \mathrm{e}\Big(\frac{m\alpha+n\delta}{c}\Big), \tag{4.3}$$

where, in the sum $\sum^*$, δ runs over the solutions $\pmod{c}$ of

$$(\delta, \gamma q/w) = 1, \quad (\gamma + u\delta, w) = 1, \quad \delta(\gamma+u\delta) \equiv u \pmod{(w,q/w)}, \tag{4.4}$$

and α is determined $\pmod{c}$ by the equations

$$\alpha\delta \equiv 1 \pmod{\gamma q/w}, \quad \alpha \equiv \gamma'\overline{u'} + u'\overline{(\gamma'+u'\delta)} \pmod{w\gamma'} \tag{4.5}$$

where $\gamma' = \gamma/(\gamma,u)$ and $u' = u/(\gamma,u)$.

The sums $S_{\mathfrak{a}\mathfrak{a}}(m,n;c)$ are expressed by means of the Chinese remainder theorem (twisted multiplicativity) as a product of similar sums for moduli c that are prime powers. When $c = p^\nu$ and $\nu \geq 2$, a bound is obtained by means of elementary methods as in [IK04, Section 12.3]. When c is prime, the Weil bound (*cf.* [KL13, Theorem 9.3]) from algebraic geometry can be used. In the general case, one obtains

Lemma 4.2. *For all $c \in \mathcal{C}(\mathfrak{a},\mathfrak{a})$, $m,n \in \mathbf{Z}$, we have*

$$S_{\mathfrak{a}\mathfrak{a}}(m,n;c) \ll (m,n,c)^{1/2}\tau(c)^{O(1)}(cq_0)^{1/2}$$

where q_0 is the modulus of χ.

Finally, we consider as in [DI82b] the following family of Kloosterman sums, which will be of particular interest to us.

Lemma 4.3. *Assume that the level q is of the shape rs, with $q_0|r$, where q_0 is the modulus of χ, and $(r,s)=1$. The two cusps ∞ and $1/s$ are singular. Choose the scaling matrices*

$$\sigma_\infty = \mathrm{Id}, \qquad \sigma_{1/s} = \begin{pmatrix} \sqrt{r} & 0 \\ s\sqrt{r} & 1/\sqrt{r} \end{pmatrix}.$$

Then $\mathcal{C}(\infty, 1/s) = \{cs\sqrt{r}, c\in\mathbf{N}, (c,r)=1\}$, and for $(c,r)=1$, we have

$$S_{\infty,1/s}(m,n;cs\sqrt{r}) = \overline{\chi}(c)\mathrm{e}\Big(\frac{n\overline{s}}{r}\Big)S(m\overline{r},n;sc)$$

where $S(\dots)$ in the right-hand side is the usual (untwisted) Kloosterman sum.

The main feature here is the presence of the character *outside* the Kloosterman sums, as opposed to (4.2). It is proven in a way identical to [DI82b, page 240], keeping track of an additional factor $\chi(D)$ in the summand.

4.1.2. *Normalization.* In order to state the Kuznetsov formula, we first fix the normalization. We largely borrow from [BHM07a]. We also refer to [DFI02, Section 4] for useful explanations on Maaß forms, and to [Pro03] for a discussion in the case of general multiplier systems.

For each integer $k>0$ with $k\equiv\kappa \pmod 2$, we fix a basis $\mathcal{B}_k(q,\chi)$ of holomorphic cusp forms. It is taken orthonormal with respect to the weight k Petersson inner product:

$$\langle f,g\rangle_k = \int_{\Gamma\backslash\mathbf{H}} y^k f(z)\overline{g(z)}\frac{\mathrm{d}x\mathrm{d}y}{y^2} \qquad (z=x+iy).$$

We let $\mathcal{B}(q,\chi)$ denote a basis of the space of Maaß cusp forms. In particular they are functions on $\mathbf{H}$, are automorphic of weight $\kappa\in\{0,1\}$ (meaning they satisfy [Pro03, formula (5)]), are square-integrable on a fundamental domain and vanish at the cusps (note that when $\kappa=1$, they do not induce a function on $\Gamma\backslash\mathbf{H}$). They are eigenfunctions of the L^2-extension of the Laplace-Beltrami operator

$$\Delta = y^2\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big) - i\kappa y\frac{\partial}{\partial x}.$$

This operator has pure point spectrum on the L^2-space of cusp forms. For $f\in\mathcal{B}(q,\chi)$, we write $(\Delta+s(1-s))f=0$ with $s=\frac12+it_f$ and $t_f\in\mathbf{R}\cup[-i/2,i/2]$. The $(t_f)_{f\in\mathcal{B}(q,\chi)}$ form a countable sequence with no limit point in $\mathbf{C}$ (in particular, there are only finitely many $t_f\in i\mathbf{R}$). We choose the basis $\mathcal{B}(q,\chi)$ orthonormal with respect to the weight zero Petersson inner product. Let

$$\theta := \sup_{f\in\mathcal{B}(q,\chi)} |\mathfrak{Im}\, t_f|, \tag{4.6}$$

then Selberg's eigenvalue conjecture is that $\theta=0$ *i.e.* $t_f\in\mathbf{R}$ for all $f\in\mathcal{B}(q,\chi)$. Selberg proved that $\theta\le 1/4$ (see [DI82b, Theorem 4]), and the current best known result is $\theta\le 7/64$, due to Kim and Sarnak [Kim03] (see [Sar95] for useful explanations on this topic).

The decomposition of the space of square-integrable, weight κ automorphic forms on $\mathbf{H}$ with respect to eigenspaces of the Laplacian contains the Eisenstein spectrum $\mathcal{E}(q,\chi)$ which turns out to be the orthogonal complement to the space of Maaß forms. It can be described explicitely by means of the Eisenstein series $E_{\mathfrak{a}}(z;\frac12+it)$ where $\mathfrak{a}$ runs through singular cusps, and $t\in\mathbf{R}$. Care must be taken because these are not square-integrable; see [IK04, Section 15.4] for more explanations.

Let $j(g,z) := cz+d$ where $g = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbf{R})$. We write the Fourier expansion of $f \in \mathcal{B}_k(q,\chi)$ around a singular cusp $\mathfrak{a}$ with associated scaling matrix $\sigma_{\mathfrak{a}}$ as

$$f(\sigma_{\mathfrak{a}} z) j(\sigma_{\mathfrak{a}}, z)^{-k} = \sum_{n \geq 1} \rho_{f\mathfrak{a}}(n)(4\pi n)^{k/2} \mathrm{e}(nz). \tag{4.7}$$

We write the Fourier expansion of $f \in \mathcal{B}(q,\chi)$ around the cusp $\mathfrak{a}$ as

$$f(\sigma_{\mathfrak{a}} z) \mathrm{e}^{-i\kappa \arg j(\sigma_{\mathfrak{a}}, z)} = \sum_{n \neq 0} \rho_{f\mathfrak{a}}(n) W_{\frac{|n|}{n}\frac{\kappa}{2}, it_f}(4\pi |n| y) \mathrm{e}(nx)$$

where the Whittaker function is defined as in [Iwa02, formula (1.26)]. Finally, for every singular cusp $\mathfrak{c}$, we write the Fourier expansion around the cusp $\mathfrak{a}$ of the Eisenstein series associated with the cusp $\mathfrak{c}$ as

$$E_{\mathfrak{c}}(\sigma_{\mathfrak{a}} z, \tfrac{1}{2}+it) \mathrm{e}^{-i\kappa \arg j(\sigma_{\mathfrak{a}}, z)} = c_{1,\mathfrak{c}}(t) y^{1/2+it} + c_{2,\mathfrak{c}}(t) y^{1/2-it} + \sum_{n \neq 0} \rho_{\mathfrak{c}\mathfrak{a}}(n,t) W_{\frac{|n|}{n}\frac{\kappa}{2}, it}(4\pi |n| y) \mathrm{e}(nx).$$

4.1.3. *The Kuznetsov formula.* Let $\phi : \mathbf{R}_+ \to \mathbf{C}$ be of class $\mathcal{C}^\infty$ and satisfy

$$\phi(0) = \phi'(0) = 0, \qquad \phi^{(j)}(x) \ll (1+x)^{-2-\eta} \quad (0 \leq j \leq 3) \tag{4.8}$$

for some $\eta > 0$. In practice, the function ϕ will be $\mathcal{C}^\infty$ with compact support in $\mathbf{R}_+^*$. We define the integral transforms

$$\dot{\phi}(k) := 4i^k \int_0^\infty J_{k-1}(x) \phi(x) \frac{\mathrm{d}x}{x}, \tag{4.9}$$

$$\widetilde{\phi}(t) := \frac{2\pi i t^\kappa}{\sinh(\pi t)} \int_0^\infty (J_{2it}(x) - (-1)^\kappa J_{-2it}(x)) \phi(x) \frac{\mathrm{d}x}{x}, \tag{4.10}$$

$$\check{\phi}(t) := 8 i^{-\kappa} \cosh(\pi t) \int_0^\infty K_{2it}(x) \phi(x) \frac{\mathrm{d}x}{x} \tag{4.11}$$

where we refer to [Iwa02, Appendix B.4] for the definitions and estimates on the Bessel functions. The sizes of these transforms is controlled by the following Lemma (we need only consider $|t| \leq 1/4$ in the second estimate, by Selberg's theorem that $\theta \leq 1/4$).

Lemma 4.4 ([DI82b, Lemma 7.1], [BHM07b, Lemma 2.1])**.** *If ϕ is supported on $x \asymp X$ with $\|\phi^{(j)}\|_\infty \ll X^{-j}$ for $0 \leq j \leq 4$, then*

$$\begin{aligned} |\dot{\phi}(t)| + \frac{|\widetilde{\phi}(t)|}{1+|t|^\kappa} + |\check{\phi}(t)| &\ll \frac{1+|\log X|}{1+X} \min\left\{1, \left(\frac{1+X}{1+|t|}\right)^3\right\} \qquad (t \in \mathbf{R}), \\ |\widetilde{\phi}(t)| + |\check{\phi}(t)| &\ll \frac{1+X^{-2|t|}}{1+X} \qquad (t \in [-i/4, i/4]). \end{aligned} \tag{4.12}$$

Proof. Taking into account the factor t^κ in front of $\widetilde{\phi}(t)$, the arguments of [DI82b, Lemma 7.1] and [BHM07b, Lemma 2.1] are easily adapted. The only non-trivial fact to check is that the decaying factor in (4.12) only requires the hypotheses $\|\phi^{(j)}\|_\infty \ll X^{-j}$ for $j \leq 4$. This is seen by reproducing the proof of [BHM07b, Lemma 2.1] with the choices $j = 1$ and $i = 2$. $\square$

Recall that κ is defined by $\chi(-1) = (-1)^\kappa$. We are ready to state the Kuznetsov formula for Dirichlet multiplier system and general cusps.

Lemma 4.5. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two singular cusps with associated scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$, and $\phi : \mathbf{R}_+ \to \mathbf{C}$ as in* (4.8)*. Let $m, n \in \mathbf{N}$. Then*

$$\sum_{c \in \mathcal{C}(\mathfrak{a},\mathfrak{b})} \frac{1}{c} S_{\mathfrak{ab}}(m, n; c) \phi\Big(\frac{4\pi\sqrt{mn}}{c}\Big) = \mathcal{H} + \mathcal{E} + \mathcal{M}, \tag{4.13}$$

$$\sum_{c \in \mathcal{C}(\mathfrak{a},\mathfrak{b})} \frac{1}{c} S_{\mathfrak{ab}}(m, -n; c) \phi\Big(\frac{4\pi\sqrt{mn}}{c}\Big) = \mathcal{E}' + \mathcal{M}', \tag{4.14}$$

where $\mathcal{H}$, $\mathcal{E}$, $\mathcal{M}$ ("holomorphic", "Eisenstein", "Maaß") are defined by

$$\mathcal{H} := \sum_{\substack{k > \kappa \\ k \equiv \kappa \pmod 2}} \sum_{f \in \mathcal{B}_k(q,\chi)} \dot{\phi}(k) \Gamma(k) \sqrt{mn}\, \overline{\rho_{f\mathfrak{a}}(m)} \rho_{f\mathfrak{b}}(n), \tag{4.15}$$

$$\mathcal{E} := \sum_{\mathfrak{c} \text{ sing.}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \tilde{\phi}(t) \frac{\sqrt{mn}}{\cosh(\pi t)} \overline{\rho_{\mathfrak{ca}}(m, t)} \rho_{\mathfrak{cb}}(n, t) \mathrm{d}t, \tag{4.16}$$

$$\mathcal{M} := \sum_{f \in \mathcal{B}(q,\chi)} \tilde{\phi}(t_f) \frac{\sqrt{mn}}{\cosh(\pi t_f)} \overline{\rho_{f\mathfrak{a}}(m)} \rho_{f\mathfrak{b}}(n), \tag{4.17}$$

$$\mathcal{E}' := \sum_{\mathfrak{c} \text{ sing.}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \check{\phi}(t) \frac{\sqrt{mn}}{\cosh(\pi t)} \overline{\rho_{\mathfrak{ca}}(m, t)} \rho_{\mathfrak{cb}}(-n, t) \mathrm{d}t, \tag{4.18}$$

$$\mathcal{M}' := \sum_{f \in \mathcal{B}(q,\chi)} \check{\phi}(t_f) \frac{\sqrt{mn}}{\cosh(\pi t_f)} \overline{\rho_{f\mathfrak{a}}(m)} \rho_{f\mathfrak{b}}(-n). \tag{4.19}$$

Proof. For $\mathfrak{a} = \mathfrak{b} = \infty$, the formula (4.13) and the case $\kappa = 0$ of (4.14) can be found in Section 2.1.4 of [BHM07a]. The extension to general cusps $\mathfrak{a}$, $\mathfrak{b}$ is straightforward.

The case $\kappa = 1$ of (4.14) was obtained by B. Topacogullari (private communications). The details are due to appear in forthcoming work, so we restrict here to mentionning that it can be proved by reproducing the computations of page 251 of [DI82b] and Section 5 of [DFI02][4]. □

The right-hand side of the Kuznetsov formula (the so-called spectral side) naturally splits into two contributions. The *regular spectrum* consists in $\mathcal{H}$, $\mathcal{E}$ and the contribution to $\mathcal{M}$ of those $f \in \mathcal{B}(q, \chi)$ with $t_f \in \mathbf{R}$; the conjecturally inexistant *exceptional spectrum* is the contribution to $\mathcal{M}$ of those f with $t_f \in i\mathbf{R}^*$ (similarly with $\mathcal{E}'$ and $\mathcal{M}'$). The technical reason for this distinction is the growth properties of the integral transforms. Indeed, when X is small (*i.e.* when the average over the moduli of the Kloosterman sums is long, since $X \asymp \sqrt{mn}/c$), we see from Lemma 4.4 that while $\dot{\phi}(t)$, $\tilde{\phi}(t)$ and $\check{\phi}(t)$ are essentially bounded for $t \in \mathbf{R}$, $\tilde{\phi}(it)$ is roughly of size $X^{-2|t|}$ when $t \in [-1/2, 1/2]$.

We remark that in contrast with other works (*e.g.* [BM15b]), we do not make use of Atkin-Lehner's newform theory, nor of Hecke theory. In fact, we do not use any information about the Fourier coefficients $\rho_{f\mathfrak{a}}(n)$ and $\rho_{\mathfrak{ca}}(n, t)$ other than the fact that Kuznetsov's formula holds, so the reader unfamiliar with the subject can go through the following sections without knowing what they are. The main feature of the Kuznetsov formula which is used is the decay properties of the integral transforms (4.9)-(4.11), and the fact that it separates the variables m and n in a way that combines very nicely with the Cauchy–Schwarz inequality.

[4] Note that in the expression for $h_p(t)$ given on page 518 of [DFI02], the term $\Gamma(1 - \frac{k}{2} - ir)$ should read $\Gamma(1 - \frac{k}{2} + ir)$.

4.2. **Large sieve inequalities.**

4.2.1. *Quadratic forms with $S_{\mathfrak{a}\mathfrak{a}}$.* Given $N \in \mathbf{N}$, $\vartheta \in \mathbf{R}_+^*$, $\lambda \geq 0$, a sequence (b_n) of complex numbers, a singular cusp $\mathfrak{a}$ and $c \in \mathcal{C}(\mathfrak{a}, \mathfrak{a})$, let

$$B_{\mathfrak{a}}(\lambda, \vartheta; c, N) := \sum_{N<m,n\leq 2N} b_m \overline{b_n} \mathrm{e}^{-\lambda\sqrt{mn}} S_{\mathfrak{a}\mathfrak{a}}(m, n, c) \mathrm{e}\Big(\frac{2\sqrt{mn}}{c}\vartheta\Big).$$

We also define

$$\|b_N\|_2^2 := \Big(\sum_{N<n\leq 2N} |b_n|^2\Big)^{1/2}.$$

The following extends [DI82b, Proposition 3].

Lemma 4.6 ([DI82b, Proposition 3])**.** *We have*

$$|B_{\mathfrak{a}}(\lambda, \vartheta; c, N)| \leq \tau(c)^{O(1)} (q_0 c)^{1/2} N \|b_N\|^2, \tag{4.20}$$

$$|B_{\mathfrak{a}}(\lambda, \vartheta; c, N)| \ll (c + N + \sqrt{\vartheta c N}) \|b_N\|^2,$$

$$|B_{\mathfrak{a}}(\lambda, \vartheta; c, N)| \ll_\varepsilon \vartheta^{-1/2} c^{1/2} N^{1/2+\varepsilon} \|b_N\|^2 \tag{4.21}$$

where the last bounds holds for $\vartheta < 2$ and $c < N$.

Proof. Suppose $\lambda = 0$. The first bound is an immediate consequence of Lemma 4.2. For the second bound, the proof given in [DI82b, page 256] transposes without any change: after expanding out the sum $S_{\mathfrak{a}\mathfrak{a}}(\dots)$, one uses the triangle inequality with the effect that the factors involving χ are trivially bounded. For the last bound, the proof is adapted with the following modification: the Cauchy–Schwarz inequality yields

$$|B_{\mathfrak{a}}(0, \vartheta; c, N)|^2 \leq \|b_N\|_2^2 \sum_{\substack{N<m_1,m_2\leq 2N \\ \delta_1,\delta_2}} b_{m_1} \overline{b_{m_2} \chi(r_1)} \chi(r_2) \mathrm{e}\Big(\frac{m_1\delta_1 - m_2\delta_2}{c}\Big) \sum_n f(n) \tag{4.22}$$

where $f(n)$ is defined as in [DI82b, page 256], δ_1 and δ_2 run over residue classes modulo c satisfying (4.4), and $r_j := \delta_j^{-1} + u(\alpha_j\delta_j - 1)/\gamma$ for $j \in \{1, 2\}$, where α_j is determined by (4.5). The only difference is the presence of the χ factors. Upon using Poisson summation on the sum $\sum_n f(n)$, the argument is split in two cases according to whether $\alpha_1 \equiv \alpha_2 \pmod c$ or not. If $\alpha_1 \not\equiv \alpha_2 \pmod c$, then one uses the triangle inequality on (4.22) so that the χ factors do not intervene. If on the contrary $\alpha_1 \equiv \alpha_2 \pmod c$, then we deduce from (4.5) that also $\delta_1 \equiv \delta_2 \pmod c$. The χ factors cancel out and the rest of the argument carries through without change.

The case of arbitrary $\lambda \geq 0$ reduces to the case $\lambda = 0$ by Mellin inversion

$$\mathrm{e}^{-y} = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \Gamma(s) y^{-s} \mathrm{d}s = 1 + \frac{1}{2\pi i} \int_{-1-i\infty}^{-1+i\infty} \Gamma(s) y^{-s} \mathrm{d}s$$

at $y = \lambda\sqrt{mn}$, using the first expression when $\lambda N \geq 1$ and the second otherwise. □

4.2.2. *Large sieve inequalities for the regular spectrum.* We proceed to state the following large sieve-type inequalities, which extend [DI82b, Proposition 4].

Proposition 4.7 ([DI82b, Proposition 4])**.** *Let (a_n) be a sequence of complex numbers, and $\mathfrak{a}$ a singular cusp for the group $\Gamma_0(q)$ and Dirichlet multiplier $\chi \pmod{q_0}$. Suppose $T \geq 1$ and $N \geq 1/2$. Then each of the three quantities*

$$\sum_{\substack{\kappa<k\leq T \\ k\equiv\kappa \pmod 2}} \Gamma(k) \sum_{f\in\mathcal{B}_k(q,\chi)} \Big| \sum_{N<n\leq 2N} a_n \sqrt{n} \rho_{f\mathfrak{a}}(n) \Big|^2, \tag{4.23}$$

$$\sum_{\substack{f\in\mathcal{B}(q,\chi)\\|t_f|\le T}} \frac{(1+|t_f|)^{\pm\kappa}}{\cosh(\pi t_f)}\Big|\sum_{N<n\le 2N} a_n\sqrt{n}\rho_{f\mathfrak{a}}(\pm n)\Big|^2, \tag{4.24}$$

$$\sum_{\mathfrak{c}\ sing.}\int_{-T}^{T}\frac{(1+|t|)^{\pm\kappa}}{\cosh(\pi t)}\Big|\sum_{N<n\le 2N} a_n\sqrt{n}\rho_{\mathfrak{c}\mathfrak{a}}(\pm n,t)\Big|^2\mathrm{d}t, \tag{4.25}$$

is majorized by

$$O_\varepsilon\Big((T^2+q_0^{1/2}\mu(\mathfrak{a})N^{1+\varepsilon})\|a_N\|_2^2\Big).$$

Here, if $\mathfrak{a}$ *is equivalent to* u/w *with* $w|q$ *and* $(u,w)=1$*, then* $\mu(\mathfrak{a}) := (w,q/w)/q$.

Proof. Theses formulas are deduced from two summation formulas, namely the Petersson formula [Iwa97, Theorem 3.6]

$$\begin{aligned} &4\Gamma(k-1)\sqrt{mn}\sum_{f\in\mathcal{B}_k(q,\chi)}\overline{\rho_{f\mathfrak{a}}(m)}\rho_{f\mathfrak{a}}(n)\\ &\qquad= \mathbf{1}_{m=n} + 2\pi i^{-k}\sum_{c\in\mathcal{C}(\mathfrak{a},\mathfrak{a})}\frac{1}{c}S_{\mathfrak{a}\mathfrak{a}}(m,n;c)J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big),\end{aligned} \tag{4.26}$$

valid for $k>1$, $k\equiv\kappa \pmod 2$, and a "pre-Kuznetsov" formula [DFI02, Proposition 5.2] which, for general cusps, is

$$\begin{aligned} &\frac{\big|\Gamma(1\mp\frac{\kappa}{2}+ir)|^2}{4\pi^2}\Big\{\mathbf{1}_{m=n}+\sum_{c\in\mathcal{C}(\mathfrak{a},\mathfrak{a})}\frac{1}{c}S_{\mathfrak{a}\mathfrak{a}}(\pm m,\pm n;c)I_\pm\Big(\frac{4\pi\sqrt{mn}}{c}\Big)\Big\}\\ &=\sum_{f\in\mathcal{B}(q,\chi)}\frac{\sqrt{mn}}{\cosh(\pi t_f)}H(t_f,r)\overline{\rho_{f\mathfrak{a}}(\pm m)}\rho_{f\mathfrak{a}}(\pm n)+\frac{1}{4\pi}\sum_{\mathfrak{c}\ \text{sing.}}\int_{-\infty}^{\infty}\frac{\sqrt{mn}}{\cosh(\pi t)}H(t,r)\overline{\rho_{\mathfrak{c}\mathfrak{a}}(\pm m)}\rho_{\mathfrak{c}\mathfrak{a}}(\pm n)\end{aligned} \tag{4.27}$$

for all real r and positive integers m, n. Here,

$$I_\pm(x) = -2x\int_{-i}^{i}(-iv)^{\pm\kappa-1}K_{2ir}(vx)\mathrm{d}v \qquad (x>0).$$

where v varies on the half-circle $|v|=1$, $\mathfrak{Re}(v)\ge 0$ counter-clockwise. Note that by the complement formula

$$\Big|\Gamma(1-\tfrac{\epsilon}{2}+ir)\Big|^2 = \frac{\pi}{\cosh(\pi r)}\times\begin{cases}1, & \epsilon=1,\\ \frac14+r^2, & \epsilon=-1.\end{cases} \tag{4.28}$$

Given the formulas (4.26) and (4.27), the arguments in [DI82b, pages 258-261] are adapted as follows. When $\kappa=0$, the details are strictly identical. Consider the case $\kappa=1$ of (4.23). We multiply both sides of (4.26) by $(k-1)\mathrm{e}^{-(k-1)/T}\overline{a_m}a_n$ and sum over k, m and n. The analog of the function $E_K(x)$ defined in [DI82b, page 258] is (up to a constant factor) the function

$$E_T(x) = \sum_{\ell\ge1}(-1)^\ell 2\ell\mathrm{e}^{-2\ell/T}J_{2\ell}(x) = -\sinh\Big(\frac1T\Big)\int_0^1\frac{u^2xJ_1(ux)\mathrm{d}u}{(\cosh(1/T)^2-u^2)^{3/2}},$$

as can be seen by reproducing the computations in [Iwa82, page 316][5]. We then write

$$J_1(y) = \frac{2}{\pi}\int_0^{\pi/2}\cos\tau\sin(y\cos\tau)\mathrm{d}\tau,$$

[5]There is a slight convergence issue in the Fourier integral for $yJ_1(y)$, which is resolved by changing $b=\cosh(1/T)$ to $b+i\varepsilon$, $\varepsilon>0$ and letting $\varepsilon\to0$.

split the integral at $\Delta \in (0, \pi/2]$ and deduce the bound (4.23) by following the steps in [DI82b, page 259].

Consider next the case $\kappa = 1$ and positive sign of (4.24) and (4.25). We multiply both sides of (4.27) by $r^2 \cosh(\pi r)\overline{a_m} a_n$, integrate over $r \in \mathbf{R}$ and sum over m and n. The analog of the function $\Phi(x)$ of [DI82b, page 260] is the function

$$\Phi_+(x) = \int_{-\infty}^{\infty} r^2 \mathrm{e}^{-(r/T)^2} \int_{-i}^{i} K_{2ir}(xv)\mathrm{d}v\mathrm{d}r.$$

We use the expression $K_{2ir}(y) = \int_0^\infty \mathrm{e}^{-y\cosh\xi}\cos(2r\xi)\mathrm{d}\xi$ $(y > 0)$. For $x > 0$, we obtain by integrations by parts

$$\begin{aligned}\Phi_+(x) &= -i\sqrt{\pi}T^3 \int_0^\infty \mathrm{e}^{-(\xi T)^2}\xi \tanh\xi \Big\{ \cos(x\cosh\xi) - \frac{1}{2}\int_{-1}^{1}\cos(x\vartheta\cosh\xi)\mathrm{d}\vartheta \Big\}\mathrm{d}\xi \\ &= -i\sqrt{\pi}\frac{T^3}{x}\int_0^\infty \mathrm{e}^{-(\xi T)^2}(1-2(\xi T)^2)\sinh(x\cosh\xi)\frac{\mathrm{d}\xi}{\cosh\xi},\end{aligned}$$

and from there, the bounds (4.24) and (4.25) are obtained by reproducing the computations of [DI82b, page 261].

Consider finally the case of negative sign in (4.24) and (4.26). We multiply both sides of (4.27) by $r^2\cosh(\pi r)/(\frac{1}{4}+r^2)\overline{a_m}a_n$. The analog of the function $\Phi(x)$ of [DI82b, page 260] is now

$$\Phi_-(x) = \int_{-\infty}^{\infty} r^2 \mathrm{e}^{-(r/T)^2}\int_{-i}^{i} K_{2ir}(xv)\frac{\mathrm{d}v}{v^2}\mathrm{d}r,$$

and we have by integration by parts

$$\begin{aligned}\Phi_-(x) &= i\sqrt{\pi}T^3\int_0^\infty \mathrm{e}^{-(\xi T)^2}\xi\tanh\xi\Big\{\cos(\xi\cosh\xi) - \frac{1}{2i}\int_{-i}^{i}\frac{\mathrm{e}^{-vx\cosh\xi}}{v^2}\mathrm{d}v\Big\}\mathrm{d}\xi \\ &= -i\sqrt{\pi}\frac{T^3}{x}\int_0^\infty \mathrm{e}^{-(\xi T)^2}(1-2(\xi T)^2)\Big\{\sinh(x\cosh\xi) + \frac{1}{i}\int_{-i}^{i}\frac{\mathrm{e}^{-xv\cosh\xi}}{v^3}\mathrm{d}v\Big\}\frac{\mathrm{d}\xi}{\cosh\xi}.\end{aligned}$$

From there, it is straightforward to reproduce the computations of [DI82b, page 261] using the bounds of Lemma 4.6.

□

4.2.3. *Weighted large sieve inequalities for the exceptional spectrum.* The objects we would like to bound now are of the shape

$$E_{q,\mathfrak{a}}(Y,(a_n)) := \sum_{\substack{f\in\mathcal{B}(q,\chi)\\ t_f\in i\mathbf{R}}} Y^{2|t_f|}\Big|\sum_{N<n\le 2N} a_n n^{1/2}\rho_{f\mathfrak{a}}(n)\Big|^2$$

where $Y \geq 1$ is to be taken as large as possible while still keeping this quantity comparable to the bounds $(1+\mu(\mathfrak{a})N)\sum_n |a_n|^2$ coming from Proposition 4.7. The following is the analog of [DI82b, Theorem 5].

Lemma 4.8. *Assume that the situation is as in Proposition 4.7. Then for any $Y \geq 1$,*

$$E_{q,\mathfrak{a}}(Y,(a_n)) \ll_\varepsilon \Big(1+(\mu(\mathfrak{a})NY)^{1/2}\Big)\Big(1+(q_0\mu(\mathfrak{a})N)^{1/2+\varepsilon}\Big)\|a_N\|_2^2.$$

The important aspect in this bound is that it is as good as those coming from the regular spectrum (*i.e.* the upper bound in Proposition 4.7) in the situation when $\mu(\mathfrak{a}) = 1/q$ (which will typically be the case), $N < q$ and $Y \leq q/N$. Note also that the previous bound holds for any individual q.

Proof. The arguments in [DI82b, section 8.1, pages 270-271] transpose identically.[6]. □

The next step is to produce an analog of [DI82b, Theorem 6], which is concerned with the situation when an average over q is done. Deshouillers and Iwaniec make use of the very nice idea that with the choice $\mathfrak{a} = \infty$ for each q, the roles of q and c can be swapped in the Kuznetsov formula. Through an induction process, this enhances significantly the bounds obtained. This switching technique is specific to the choice $\mathfrak{a} = \infty$ for all q, with scaling matrices independent of q.

Lemma 4.9. *Assume the situation is as previously. Recall that χ has modulus $q_0 \geq 1$. Then for all $Y \geq 1$ and $Q \geq q_0$,*

$$\sum_{\substack{q \leq Q \\ q_0 | q}} E_{q,\infty}(Y, (a_n)) \ll_\varepsilon (QN)^\varepsilon (Qq_0^{-1} + N + NY^{1/2}) \|a_N\|_2^2,$$

where the scaling matrices are chosen independently of q.

Note that now, in the situation when $N \leq Q$, the parameter Y is allowed to be as large as $(Q/N)^2$ while still yielding a bound of same quality as the regular spectrum. The final situation is the special case when (a_n) is the characteristic sequence of an interval of integers. Then Deshouillers and Iwaniec are able to provide an even stronger bound [DI82b, Theorem 7], by enhancing the initial step in the induction.

Lemma 4.10. *Assume that the situation is as in Lemma 4.9. Assume moreover that $(a_n)_{N<n\leq 2N}$ is the characteristic sequence of an interval of integers. Then*

$$\sum_{\substack{q \leq Q \\ q_0 | q}} E_{q,\infty}(Y, (a_n)) \ll_\varepsilon (QN)^\varepsilon (Qq_0^{-1} + N + (NY)^{1/2}) N.$$

In the situation when $N \leq Q$, the parameter Y can then be taken as large as Q^2/N while still yielding an acceptable bound.

We now proceed to justify Lemmas 4.9 and 4.10. For the rest of this section, we rename q into $q_0 q$, so that now q runs over intervals. The object of interest is

$$S(Q, Y, N, s) := \sum_{Q < q \leq 16Q} \sum_{\substack{f \in \mathcal{B}(q_0 q, \chi) \\ t_f \in i\mathbf{R}}} Y^{2|t_f|} \Big| \sum_{N < n \leq 2N} a_n n^{s+1/2} \rho_{f\infty}(n) \Big|^2.$$

Lemma 4.11. *Let $N, Y, Q \geq 1$ and a sequence (a_n) be given. Then*

$$\begin{aligned} S(Q, Y, N, 0) \ll_\varepsilon & \int_{-\infty}^{\infty} S\Big(\frac{\pi NY}{q_0 Q}, Y, N, it\Big) \frac{\mathrm{d}t}{t^4 + 1} \\ & + (QYN)^\varepsilon \Big(Q + \frac{N}{q_0^{1/2}} + \frac{NY}{q_0^{1/2} Q}\Big) \|a_N\|_2^2. \end{aligned} \tag{4.29}$$

Moreover, if (a_n) is the characteristic sequence of an interval, then

$$S(Q, Y, N, 0) \ll_\varepsilon (NY)^\varepsilon (Q + N + Y) N \tag{4.30}$$

Proof of (4.29). The arguments in [DI82b, pages 272-273] are adapted with minimal effort; however we take the opportunity to justify more precisely one of the claims made there. Fix a smooth function $\Phi : \mathbf{R} \to [0, 1]$ supported inside $[1/2, 5/2]$ and majorizing $\mathbf{1}_{[1,2]}$. Letting $g(q) = \Phi(q/Q)$ and $\phi(x) = \Phi(Yx)$ (these kind of homotheties of Φ we refer to as *test functions*) we have

$$S(Q, Y, N, 0) \ll |\mathcal{S}_1|,$$

[6] Note that in the last display of the proof [DI82b, page 271], $L(Y)$ should read $L(Y^{-1})$.

$$\mathcal{S}_1 := \sum_{q\geq 1} g(q) \sum_{\substack{f\in\mathcal{B}(q_0q,\chi)\\ t_f\in i\mathbf{R}}} \frac{\widetilde{\phi}(t_f)}{\cosh(\pi t_f)} \Big| \sum_n a_n n^{1/2}\rho_{f\infty}(n)\Big|^2.$$

This is seen by approximating the Bessel function in the definition of $\widetilde{\phi}$ by its first order term, as in [DI82b, formula (8.1)]. Opening the squares in $\mathcal{S}_1$ and applying the Kuznetsov formula and the large sieve estimates (Lemma 4.5 and Proposition 4.7), one gets

$$\mathcal{S}_1 = \sum_{m,n} \overline{a_m} a_n \mathcal{S}_2(m,n) + O_\varepsilon\Big((QNY)^\varepsilon\Big(Q + \frac{N}{q_0^{1/2}}\Big)\sum_n |a_n|^2\Big),$$

$$\mathcal{S}_2(m,n) := \sum_{q,c\geq 1} \frac{g(q)}{q_0qc}\phi\Big(\frac{4\pi\sqrt{mn}}{q_0qc}\Big)S_{\infty\infty}(m,n;qc),$$

Letting $h(x) = h_{m,n,c}(x) = \phi(x)g\Big(\frac{4\pi\sqrt{mn}}{q_0cx}\Big)$, one applies the Kuznetsov formula for the group $\Gamma_0(q_0c)$ (which requires that the scaling matrices be independent of q) and obtains

$$\mathcal{S}_1 \ll |\mathcal{S}_3| + O_\varepsilon\Big((QNY)^\varepsilon\Big(Q + \frac{N}{q_0^{1/2}} + \frac{NY}{q_0^{1/2}Q}\Big)\sum_n |a_n|^2\Big),$$

$$\mathcal{S}_3 := \sum_{m,n} \overline{a_m} a_n \sum_{C<c\leq 16C} \sum_{\substack{f\in\mathcal{B}(q_0c,\chi)\\ t_f\in i\mathbf{R}}} \frac{\widetilde{h}(t_f)}{\cosh(\pi t_f)} \sqrt{mn}\,\overline{\rho_{f\infty}(m)}\rho_{f\infty}(n).$$

Note that $h(t_f) = h_{m,n,c}(t_f) = 0$ unless $C < c \leq 16C$, where $C = \pi NY/(q_0Q)$. Let

$$\mathcal{K}_{\kappa,t}(x) := \frac{2\pi i t^\kappa}{\sinh(\pi t)}\Big(J_{2it}(x) - (-1)^\kappa J_{-2it}(x)\Big),$$

and $\breve{g}(s) := \int_0^\infty g(x)x^{s-1}\mathrm{d}x$ be the Mellin transform of g. Then

$$\widetilde{h}(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \breve{g}(i\tau)\Big(\frac{q_0c}{4\pi\sqrt{mn}}\Big)^{i\tau}\int_0^\infty \mathcal{K}_{\kappa,t}(x)x^{i\tau}\phi(x)\mathrm{d}x\mathrm{d}\tau.$$

Inserting into the definition of $\mathcal{S}_3$ and using the triangle inequality, we obtain

$$\mathcal{S}_3 \ll \int_{-\infty}^\infty |\breve{g}(i\tau)| \sum_{C<c\leq 16C} \sum_{\substack{f\in\mathcal{B}(q_0c,\chi)\\ t_f\in i\mathbf{R}}} \Big|\sum_m a_m m^{(1+i\tau)/2}\rho_{f\infty}(m)\Big|\Big|\sum_n a_n n^{(1-i\tau)/2}\rho_{f\infty}(m)\Big| \times$$
$$\times \Big|\int_0^\infty \mathcal{K}_{\kappa,t}(x)x^{i\tau}\phi(x)\mathrm{d}x\Big|\mathrm{d}\tau.$$

From there, the arguments in [DI82b, page 273] apply and yield

$$\Big|\int_0^\infty \mathcal{K}_{\kappa,t}(x)x^{i\tau}\phi(x)\mathrm{d}x\Big| \ll_\varepsilon Y^{2|t_f|} + Y^\varepsilon$$

from which the claimed bound follows in the same way as [DI82b, page 273]. □

Proof of (4.30). Assume that $(a_n)_{N<n\leq 2N}$ is the characteristic sequence of the integers inside $(N, N_1]$ for some $N_1 \leq 2N$. We proceed as in [DI82b, page 276]. By applying the Kuznetsov formula and the large sieve inequalities, one obtains

$$S(Q,N,Y,0) \ll_\varepsilon \sum_{Q<q\leq 16Q}\sum_{c\geq 1}\frac{1}{q_0qc}\Big|\sum_{N\leq m,n\leq N_1}\phi\Big(\frac{4\pi\sqrt{mn}}{q_0qc}\Big)S_{\infty\infty}(m,n;qc)\Big|$$
$$+ \Big(Q + \frac{N^{1+\varepsilon}}{q_0^{1/2}}\Big)N$$

for a test function ϕ supported inside $[1/(2Y), 5/(2Y)]$. Here one may restrict summation to $C/4 < c \leq 8C$ for $C := \pi NY/(q_0 Q)$. Let $k := q_0 qc$. The first term above is majorized by

$$T := (q_0 QC)^{-1+\varepsilon} \sum_{\substack{k \asymp q_0 QC \\ q_0 | k}} \Big| \sum_{N < m,n \leq N_1} \phi\Big(\frac{4\pi\sqrt{mn}}{k}\Big) S_{\infty\infty}(m, n; kq_0^{-1}) \Big|.$$

Let $\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \breve{\phi}(it)x^{-it}\mathrm{d}t$, where the Mellin transform $\breve{\phi}(s) = \int_0^\infty \phi(x)x^{s-1}\mathrm{d}x$ satisfies $\breve{\phi}(it) \ll (1+t^4)^{-1}$, so that (after reinterpreting t by $2t$)

$$T \ll (q_0 QC)^{-1+\varepsilon} \int_{-\infty}^{\infty} \frac{1}{t^4+1} \sum_{\substack{k \asymp q_0 QC \\ q_0 | k}} \Big| \sum_{N < m,n \leq N_1} (mn)^{-it} \mathrm{e}((m-n)\vartheta) S_\chi(m, n; k) \Big| \mathrm{d}t$$

for some $\vartheta \in [0,1)$ (depending on the scaling matrix). By $m^{-it} = N_1^{-it} + it\int_m^{N_1} u^{-it-1}\mathrm{d}u$, we obtain

$$T \ll (q_0 QC)^{-1+\varepsilon} \sup_{N \leq N', M' \leq N_1} \sum_{\substack{k \asymp q_0 QC \\ q_0 | k}} U_1(k, M', N'),$$

$$U_1(M', N') := \Big| \sum_{\substack{m \leq M' \\ n \leq N'}} \mathrm{e}((m-n)\vartheta) S_\chi(m, n; k) \Big|.$$

Opening the summation in S_χ, we have

$$U_1(k, M', N') \leq U_2(k, M', N') := \sum_{\delta \pmod{k}^\times} \Big| \sum_{m \leq M'} \mathrm{e}\Big(\frac{\delta m}{k} + m\vartheta\Big) \Big| \Big| \sum_{n \leq N'} \mathrm{e}\Big(\frac{\overline{\delta} n}{k} - n\vartheta\Big) \Big|.$$

It is crucial to note that the quantity on the RHS also exists for k not multiple of q_0, so trivially

$$T \ll (q_0 QC)^{-1+\varepsilon} \sup_{N \leq M', N' \leq N_1} \sum_{k \asymp q_0 QC} U_2(k, M', N'),$$

From there on, the calculations in [DI82b, page 276] apply and yield, in the notation of [DI82b, Lemma 8.2],

$$U_2(k, M', N') \ll \sum_{m,n \in \mathbf{Z}} \widehat{f}_{M'}(m)\mathrm{e}(m\vartheta)\widehat{f}_{N'}(n)\mathrm{e}(-n\vartheta)S(m, n; k).$$

The proof of Theorem 14 of [DI82b] follows through, and yields for all $K \geq 1$,

$$\sum_{k \leq K} U_2(k, M', N') \ll_\varepsilon (KMN)^\varepsilon K(K + MN).$$

Taking $K \asymp q_0 QC$, we conclude that

$$T \ll_\varepsilon (q_0 QC)^\varepsilon (q_0 QC + N^2).$$

The rest of the arguments in [DI82b, page 277] applies and yields

$$S(Q, N, Y, 0) \ll_\varepsilon (NY)^\varepsilon (Q + N + Y)N$$

as claimed. □

Proof of Lemmas 4.9 and 4.10. In addition to the recurrence relation (4.29), we have the properties

$$S(Q,Y,N,0) \leq (Y/Z)^{1/2} S(Q,Z,N,0) \qquad (1 \leq Z \leq Y),$$
$$S(Q,1,N,0) \ll_\varepsilon (QN)^\varepsilon \Big(Q + \frac{N}{q_0^{1/2}}\Big) \|a_N\|_2^2.$$

The second one follows from Proposition 4.7. Having these at hand, the induction arguments in [DI82b, page 274] and [DI82b, page 277] are easily reproduced. It is useful to notice that q_0 appears only with negative powers in the error terms, and that its presence in the denominator of $\pi NY/(q_0 Q)$ in (4.29) is beneficial for the induction. □

Remark. The previous three lemmas used only Selberg's theorem that $\theta \leq 1/4$ (recall the definition (4.6)). One could make the bounds explicit in terms of θ and thus benefit from recent progress towards the Ramanujan-Selberg conjecture. It is straightforward to check that Lemmas 4.8, 4.9 and 4.10 hold with the right-hand sides replaced by

$$(1 + (\mu(\mathfrak{a})NY)^{2\theta})(1 + q_0^{1/2}(\mu(\mathfrak{a})N)^{1-2\theta+\varepsilon})\|a_N\|_2^2,$$
$$(QN)^\varepsilon (Qq_0^{-1} + N + Y^{2\theta} N^{4\theta} Q^{1-4\theta})\|a_N\|_2^2,$$
$$(QN)^\varepsilon (Qq_0^{-1} + N + Y^{2\theta} N^{2\theta} Q^{1-4\theta}) N$$

respectively (compare with [IK04, Proposition 16.10]). We refrain from doing so because it would not impact the applications considered here.

4.3. **Proof of Theorem 2.1.**

4.3.1. *Estimates for sums of generalized Kloosterman sums.* We begin by the following statement regarding the generalized Kloosterman sums $S_{\mathfrak{a},\mathfrak{b}}(m,n;c)$. For the sake of simplifying the presentation of the bound obtained, we discard powers of the modulus q. This does not have consequences on our applications.

Proposition 4.12. *Let the real numbers $M,N,R,S \geq 1$, $X > 0$ and the integer $q \geq 1$ be given, let χ be a character modulo q, let ϕ be a smooth function supported on the interval $[X, 2X]$ such that $\|\phi^{(j)}\|_\infty \ll X^{-j}$ for $0 \leq j \leq 4$, and let (a_m) and $(b_{n,r,s})$ be sequences of complex numbers supported on $M < m \leq 2M$, $N < n \leq 2N$, $R < r \leq 2R$ and $S < s \leq 2S$. Assume that (a_m) is the characteristic sequence of an interval of integers. Then*

$$\sum_{\substack{m,n,r,s\\(s,rq)=1}} a_m b_{n,r,s} \sum_{c \in \mathcal{C}(\infty,1/s)} \frac{1}{c}\phi\Big(\frac{4\pi\sqrt{mn}}{c}\Big) S_{\infty,1/s}(m,\pm n;c) \tag{4.31}$$
$$\ll_\varepsilon (q(X+X^{-1})RSMN)^\varepsilon \Big\{L_{\rm reg} + L_{\rm exc}\Big\},$$

$$L_{\rm reg} := \Big(1 + X + \sqrt{\frac{N}{RS}}\Big)\Big(1 + X + \sqrt{\frac{M}{RS}}\Big)\frac{\sqrt{RS}}{1+X}\sqrt{M}\|b_{N,R,S}\|_2,$$
$$L_{\rm exc} := \Big(1 + \sqrt{\frac{N}{RS}}\Big)\sqrt{\frac{1+X^{-1}}{RS}}\Big(\frac{MN}{RS+N}\Big)^{1/4}\frac{\sqrt{RS}}{1+X}\sqrt{M}\|b_{N,R,S}\|_2.$$

where the Kloosterman sum is defined with respect to the congruence group $\Gamma(qrs)$ with multiplier induced by χ, with scaling matrices σ_∞ and $\sigma_{1/s}$ that are both independent of m and n, with σ_∞ independent of r and s as well.

Remark. If (a_m) is not the characteristic sequence of an interval, then the bound (4.31) still holds with $L_{\rm exc}$ is replaced by $M^{1/4}L_{\rm exc}$ (see [DI82b, Theorems 10 and 11]).

Proof. This estimate is deduced from Proposition 4.7 and Lemmas 4.8 and 4.10 by following the computations of Section 9.1 of [DI82b]. It is useful to notice that the bounds of Lemmas 4.8, 4.10 and Proposition 4.7 (for $\mathfrak{a} \in \{\infty, 1/s\}$) decrease with q_0. $\square$

4.3.2. *Estimates for the complete Kloosterman sums twisted by a character.* We now justify the transition from Proposition 4.12 to an estimate for twisted sums of usual Kloosterman sums $S(m, n; c)$.

Proposition 4.13. *Let the real numbers $M, N, R, S, C \geq 1$, and the integer $q \geq 1$ be given, let χ be a character modulo q, let g be a smooth function supported on $[C, 2C] \times [M, 2M] \times (\mathbf{R}_+^*)^3$ such that*

$$\frac{\partial^{\nu_0+\nu_1+\nu_2+\nu_3+\nu_4} g}{\partial c^{\nu_0}\partial m^{\nu_1}\partial n^{\nu_2}\partial r^{\nu_3}\partial s^{\nu_4}}(c, m, n, r, s) \ll C^{-\nu_0}M^{-\nu_1}N^{-\nu_2}R^{-\nu_3}S^{-\nu_4} \tag{4.32}$$

for $0 \leq \nu_j \leq 12$. Let $(b_{n,r,s})$ be a sequence of complex numbers supported on $N < n \leq 2N$, $R < r \leq 2R$ and $S < s \leq 2S$. Then uniformly in $t \in [0, 1)$,

$$\begin{aligned}\sum_{\substack{c,m,n,r,s\\(sc,rq)=1}} b_{n,r,s}\ \chi(c) g(c, m, n, r, s) \mathrm{e}(mt) S(n\overline{r}, \pm m\overline{q}; sc)\\ \ll_\varepsilon (CRSMN)^\varepsilon q^{3/2}\Big\{K_{\rm reg} + K_{\rm exc}\Big\}\sqrt{M}\|b_{N,R,S}\|,\end{aligned} \tag{4.33}$$

$$K_{\rm reg}^2 := RS\frac{(C^2S^2R + MN + C^2SN)(C^2S^2R + MN + C^2SM)}{C^2S^2R + MN},$$

$$K_{\rm exc}^2 := C^3S^2\sqrt{R(N + RS)}.$$

Proof. As before, we present the proof in the case where there is a + sign in the Kloosterman sums. The complementary case is similar. The main issue is separation of variables, as explained in [DI82b, page 269]. The nuisance is mainly notational. We write

$$g(c, m, n, r, s) = \int_{\mathbf{R}^4} \tfrac{1}{sc\sqrt{rq}} G\big(\tfrac{4\pi\sqrt{mn}}{sc\sqrt{rq}}, \xi_1, \xi_2, \xi_3, \xi_4\big)\mathrm{e}(-m\xi_1 - n\xi_2 - r\xi_3 - s\xi_4)\prod_{j=1}^4 \mathrm{d}\xi_j,$$

by Fourier inversion, where for all $(x, \xi_1, \dots, \xi_4) \in \mathbf{R}_+^* \times \mathbf{R}^4$,

$$G(x, \xi_1, \dots, \xi_4) := \int_{(\mathbf{R}_+^*)^4} g_*(x, x_1, \dots, x_4)\mathrm{e}(x_1\xi_1 + \dots + x_4\xi_4)\prod_{j=1}^4 \mathrm{d}x_j,$$

$$g_*(x, x_1, \dots, x_4) := \frac{4\pi\sqrt{x_1x_2}}{x} g\Big(\frac{4\pi\sqrt{x_1x_2}}{xx_4\sqrt{x_3q}}, x_1, \dots, x_4\Big).$$

By integration by parts, for any non-negative integers $(\ell, \ell_1, \dots, \ell_4)$ with $\ell \leq 4$ and $\ell_j \leq 2$,

$$\begin{aligned}\frac{\partial^\ell G}{\partial x^\ell}(x, \xi_1, \dots, \xi_4) = \prod_j (2\pi i\xi_j)^{-\ell_j}\int_{\mathbf{R}^4}\Big(\frac{\partial^{\ell+\ell_1+\dots+\ell_4}}{\partial x^\ell\partial x_1^{\ell_1}\cdots\partial x_4^{\ell_4}} g_*(x, x_1, \dots, x_4)\Big)\times\\ \times\, \mathrm{e}(x_1\xi_1 + \dots + x_4\xi_4)\prod_j \mathrm{d}x_j\end{aligned}$$

assuming $\xi_j \neq 0$ if $\ell_j > 0$. The derivatives are estimated using (4.32). Choose $\ell_1 = 0$ or $\ell_1 = 2$ according to whether $|\xi_1|M < 1$ or not, and similarly for ℓ_2, ℓ_3, ℓ_4. Then

$$\frac{\partial^\ell G}{\partial x^\ell}(x, \xi_1, \dots, \xi_4) \ll \frac{MNRS^2C\sqrt{qR}(\sqrt{MN}/(CS\sqrt{qR}))^{-\ell}}{(1+(\xi_1 M)^2)(1+(\xi_2 N)^2)(1+(\xi_3 R)^2)(1+(\xi_4 S)^2)}.$$

We abbreviate further

$$\phi(x) = \phi_{\xi_1,\dots,\xi_4}(x) := \frac{(1+(\xi_1 M)^2)(1+(\xi_2 N)^2)(1+(\xi_3 R)^2)(1+(\xi_4 S)^2)}{MNRS^2C\sqrt{qR}} G(x, \xi_1, \dots, \xi_4).$$

This function satisfies the hypotheses of Proposition 4.12, with[7] $X = \sqrt{MN}/(CS\sqrt{qR})$, uniformly in ξ_j. Define

$$\widetilde{b}_{n,r,s} := b_{n,r,s}\mathrm{e}\big(n(\xi_2 + \overline{s}/(rq)) - r\xi_3 - s\xi_4\big).$$

Finally, by Lemma 4.3 with an appropriate choice of scaling matrix (depending on ξ_1 and t), we have

$$\chi(c)S(n\overline{r}, m\overline{q}; sc)\mathrm{e}(m(t-\xi_1)) = S_{\infty,1/s}(m,n; sc\sqrt{rq}).$$

Proposition 4.12 can therefore be applied and yields

$$\begin{aligned}\sum_{\substack{m,n,r,s\\(s,rq)=1}} \widetilde{b}_{n,r,s} \sum_{(c,rq)=1} \frac{1}{cs\sqrt{rq}}\, \phi\Big(\frac{4\pi\sqrt{mn}}{sc\sqrt{rq}}\Big) S_{\infty,1/s}(m,n;sc\sqrt{rq}) \\ \ll_\varepsilon \frac{q^{3/2}(CMNRS)^\varepsilon}{CS\sqrt{qR}}(W_{\mathrm{reg}} + W_{\mathrm{exc}})\sqrt{M}\|b_{N,R,S}\|_2,\end{aligned}$$

with

$$\begin{aligned} W_{\mathrm{reg}}^2 &= RS\frac{(C^2S^2R + MN + C^2SN)(C^2S^2R + MN + C^2SM)}{C^2S^2R + MN}, \\ W_{\mathrm{exc}}^2 &= C^3S^2\sqrt{R(N+RS)}. \end{aligned}$$

From the definitions of ϕ and G, we deduce the claimed bound. □

4.3.3. *Bounds for incomplete Kloosterman sums.* In this section, we prove Theorem 2.1. As a first reduction, we remark that it suffices to prove the result when the sequence $b_{n,r,s}$ is supported on $N < n \leq 2N$, by summing dyadically over N and by concavity of $\sqrt{\cdot}$ (losing a factor $(\log N)^{1/2}$ in the process). Secondly, we let $s_0 \pmod q^\times$ be fixed and assume without loss of generality that

$$b_{n,r,s} = 0 \text{ unless } s \equiv s_0 \pmod q. \tag{4.34}$$

We will recover the full bound (2.3) by summing over $s_0 \pmod q^\times$ (losing a factor $q^{1/2}$ in the process by concavity). Let

$$\ddot{g}(c,m,n,r,s) := \int_{-\infty}^{\infty} g(c,\xi,n,r,s)\mathrm{e}(\xi m)\mathrm{d}\xi. \tag{4.35}$$

[7]Note that in [DI82b, page 278], some occurences of X should read X^{-1}.

By Poisson summation, we write the left-hand side of (2.3) as

$$\begin{aligned}
&\sum_{\substack{c,n,r,s\\(qr,sc)=1\\c\equiv c_0\pmod q}} b_{n,r,s} \sum_{\substack{\delta\pmod{sc}\\(\delta,sc)=1}} \mathrm{e}\Big(n\frac{\overline{r\delta}}{sc}\Big) \sum_{\substack{d\equiv\delta\pmod{sc}\\d\equiv d_0\pmod q}} g(c,d,n,r,s)\\
&= \sum_{\substack{c,n,r,s\\(qr,sc)=1\\c\equiv c_0\pmod q}} \frac{b_{n,r,s}}{scq} \sum_{(\delta,sc)=1} \mathrm{e}\Big(n\frac{\overline{r\delta}}{sc}\Big) \sum_m \ddot g(c,m/sqc,n,r,s)\mathrm{e}\Big(-\frac{md_0\overline{sc}}{q}-\frac{m\delta\overline{q}}{sc}\Big)\\
&= \sum_{\substack{c,m,n,r,s\\(qr,sc)=1\\c\equiv c_0\pmod q}} \frac{b_{n,r,s}}{scq}\ddot g(c,m/scq,n,r,s)\mathrm{e}\Big(\frac{-md_0\overline{s_0c_0}}{q}\Big)S(n\overline{r},-m\overline{q};sc)
\end{aligned}\tag{4.36}$$

where $S(\dots)$ is the usual Kloosterman. Let $M>0$ be a parameter. We write (4.36) as $\mathcal{A}_0+\mathcal{A}_\infty+\mathcal{B}$, where $\mathcal{A}_0$ is the contribution of $m=0$, $\mathcal{A}_\infty$ is the contribution of indices m such that $|m|>M$, and $\mathcal{B}$ is the contribution of indices m with $0<|m|\le M$. By the bound for Ramanujan sums [IK04, formula (3.5)],

$$\mathcal{A}_0 \ll \frac1q \sum_{\substack{c,n,r,s\\(qr,sc)=1\\c\equiv c_0\pmod q}} \frac{|b_{n,r,s}|}{sc}|\ddot g(c,0,n,r,s)|(n,sc) \ll q^{-2}(\log S)^2 D\{NR/S\}^{1/2}\|b_{N,R,S}\|_2.$$

By repeated integration by parts in the integral (4.35), for fixed $k\ge1$ and $m\ne0$ we have

$$\ddot g(c,m/(scq),n,r,s) \ll_k D^{1-k(1-\varepsilon_0)}\Big(\frac{scq}{|m|}\Big)^k.$$

Taking $k\asymp 1/\varepsilon_0$, we have that there is a choice of $M\ll (SCqD)^{\varepsilon+O(\varepsilon_0)}SCq/D$ such that the bound

$$\ddot g(c,m/(scq),n,r,s) \ll_\varepsilon 1/m^2 \qquad (|m|>M)$$

holds. Bounding trivially the Kloosterman sum in (4.36) by sc, we obtain

$$\mathcal{A}_\infty \ll_\varepsilon (SCqD)^{\varepsilon+O(\varepsilon_0)}q^{-2}D\{NR/S\}^{1/2}\|b_{N,R,S}\|_2 \tag{4.37}$$

which is also acceptable (if ε_0 is small enough, the factor $q^{-2+\varepsilon+O(\varepsilon_0)}$ is bounded).

There remains to bound $\mathcal{B}$; we may assume that $M\ge1$ for otherwise $\mathcal{B}$ is void. By dyadic decomposition,

$$|\mathcal{B}| \ll \log M \sup_{1/2\le M_1\le M} |\mathcal{B}(M_1)|,$$

where

$$\mathcal{B}(M_1) := \sum_{\substack{c,m,n,r,s\\(qr,sc)=1\\M_1<|m|\le 2M_1\\c\equiv c_0\pmod q}} \frac{b_{n,r,s}}{scq}\ddot g(c,m/scq,n,r,s)\mathrm{e}\Big(\frac{-md_0\overline{s_0c_0}}{q}\Big)S(n\overline{r},-m\overline{q};sc).$$

We insert the definition of $\ddot g$ after having changed variables $\xi\to\xi scq/m$, to obtain

$$|\mathcal{B}(M_1)| \ll \frac{DM_1}{SCq} \sup_{\xi\asymp DM_1/(SQq)} |\mathcal{B}'(M_1,\xi)|,$$

where

(4.38)
$$\mathcal{B}'(M_1,\xi) := \sum_{\substack{c,m,n,r,s\\(qr,sc)=1\\M_1<|m|\le 2M_1\\c\equiv c_0 \pmod q}} \frac{b_{n,r,s}}{m} g(c,\xi scq/m,n,r,s)\mathrm{e}\Big(\frac{-md_0\overline{s_0c_0}}{q}\Big)S(n\overline{r},-m\overline{q};sc).$$

By orthogonality of multiplicative characters, we have

$$\mathcal{B}'(M_1,\xi) = \frac{1}{M_1\varphi(q)}\sum_{\chi \pmod q}\chi(c_0)\mathcal{S}(M_1,\xi,\chi),$$

where

$$\mathcal{S}(M_1,\xi,\chi) := \sum_{\substack{r,s\\(s,qr)=1}}\sum_{\substack{m,n\\|m|\asymp M_1}} b_{n,r,s}\sum_{(c,rq)=1}\overline{\chi(c)}g_1(c,m,n,r,s)\mathrm{e}\Big(\frac{-md_0\overline{s_0c_0}}{q}\Big)S(n\overline{r},-m\overline{q};sc),$$

$$g_1(c,m,n,r,s) := M_1m^{-1}g(c,\xi scq/m,n,r,s).$$

Proposition 4.13 can be applied to the sums $\mathcal{S}(M_1,\xi,\chi)$, at the cost of enlarging the bound by a factor $O((CDNRS)^{60\varepsilon_0})$ in order for the derivative conditions (4.32) to be satisfied. We obtain

$$\mathcal{S}(M_1,\xi,\chi) \ll_\varepsilon q^{3/2}(CDNRS)^{\varepsilon+O(\varepsilon_0)}\Big\{L_{\mathrm{reg}}+L_{\mathrm{exc}}\Big\}\sqrt{M_1}\|b_{N,R,S}\|_2,$$

$$L_{\mathrm{reg}}^2 := RS\frac{(C^2S^2R+M_1N+C^2SN)(C^2S^2R+M_1N+C^2SM_1)}{C^2S^2R+M_1N},$$

$$L_{\mathrm{exc}}^2 := C^3S^2\sqrt{R(N+RS)}.$$

From there, computations identical to [DI82b, page 282] allow to bound

$$L_{\mathrm{reg}} \ll RS\Big(C^2S^2R+M_1N+\frac{C^2M_1N}{R}+C^2S(M_1+N)\Big).$$

We deduce successively

$$|\mathcal{B}(M_1)| \ll_\varepsilon (CDNRS)^{\varepsilon+O(\varepsilon_0)}\frac{q^{1/2}D\sqrt{M_1}}{SC}L^*(M_1)\|b_{N,R,S}\|_2,$$

$$L^*(M_1)^2 := RS(C^2S^2R+M_1N+C^2M_1N/R+C^2S(M_1+N))+C^3S^2\sqrt{R(N+RS)},$$

and finally

(4.39)
$$\mathcal{B} \ll_\varepsilon (CDNRS)^{\varepsilon+O(\varepsilon_0)}q^{1/2}\mathcal{K},$$

$$\mathcal{K}^2 := CS(N+RS)(C+RD)+C^2DS\sqrt{(N+RS)R}.$$

Grouping our two bounds (4.37) and (4.39), and summing over $s_0 \pmod q^\times$, we obtain the claimed result.

5. Convolutions in arithmetic progressions

In this section, we proceed with an instance of the dispersion method, for convolutions of two sequences one of which is supported in $[x^\eta, x^{1/3-\eta}]$ for some $\eta > 0$. This extends [BFI86, Section 13] and [Fou85, Section V].

Given a parameter $R \geq 1$, an integer $q \geq 1$ and a residue class $n \pmod q$, we let

$$\mathcal{X}_q(R) := \{\chi \pmod q,\ \mathrm{cond}(\chi) \leq R\},$$

and

$$\begin{aligned} \mathfrak{u}_R(n;q) &:= \mathbf{1}_{n\equiv 1 \pmod q} - \frac{1}{\varphi(q)} \sum_{\chi \in \mathcal{X}_q(R)} \chi(n) \\ &= \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \mathrm{cond}(\chi) > R}} \chi(n). \end{aligned} \tag{5.1}$$

Note that this vanishes when $q \leq R$. We have the trivial bound

$$|\mathfrak{u}_R(n;q)| \ll \mathbf{1}_{n\equiv 1 \pmod q} + \frac{R\tau(q)}{\varphi(q)}. \tag{5.2}$$

It will also be sometimes useful to write

$$\mathfrak{u}_R(n;q) = \Big(\mathbf{1}_{n\equiv 1 \pmod q} - \frac{\mathbf{1}_{(n,q)=1}}{\varphi(q)}\Big) - \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ 1<\mathrm{cond}(\chi)\leq R}} \chi(n). \tag{5.3}$$

Theorem 5.1. *Let M, N, Q, $R \geq 1$ and η be given, with $x := MN$ and $x^{1/4} \leq Q$. Then there exists δ depending at most on η such that the following holds. Let two sequences (α_m), (β_n) supported in $n \in (N, 2N]$ and $m \in (M, 2M]$ be given, which satisfy for some $A \geq 1$,*

$$|\alpha_m| \leq \tau(m)^A, \qquad |\beta_n| \leq \tau(n)^A. \tag{5.4}$$

Let $a_1, a_2 \in \mathbf{Z} \smallsetminus \{0\}$, and assume that

$$\begin{cases} x^\eta \leq N \leq Q^{2/3-\eta}, \\ Q \leq x^{1/2+\delta}, \\ R, |a_1|, |a_2| \leq x^\delta. \end{cases} \tag{5.5}$$

Then for small enough η, we have

$$\sum_{\substack{Q<q\leq 2Q \\ (q,a_1a_2)=1}} \sum_{\substack{n,m \\ (n,a_2)=1}} \alpha_m\beta_n \mathfrak{u}_R(mn\overline{a_1}a_2;q) \ll x(\log x)^{O(1)}R^{-1}. \tag{5.6}$$

The implicit constants depend on η and A at most.

Introducing $\mathfrak{u}_R(n;q)$ is technically much more convenient than the usual

$$\mathfrak{u}_1(n;q) = \mathbf{1}_{n\equiv 1 \pmod q} - \frac{\mathbf{1}_{(n,q)=1}}{\varphi(q)}. \tag{5.7}$$

Indeed, there are no equidistribution assumptions on our sequences in Theorem 5.1.

5.1. **Bombieri-Vinogradov range.** Before we embark on the dispersion method we need an estimate which is relevant to values of the moduli less than the threshold $x^{1/2-\varepsilon}$.

Lemma 5.2. *Let $M, N, R \geq 1$. Let $x = MN$, and suppose we are given two sequences (α_m) and (β_n) supported on the integers of $(M, 2M]$ and $(N, 2N]$ respectively, satisfying the bounds* (5.4)*. Suppose that $Q \leq x^{1/2}/R$ and $R \leq Q$. Then*

$$\sum_{Q<q\leq 2Q} \max_{\substack{0<a<q \\ (a,q)=1}} \Big| \sum_{m,n} \alpha_m \beta_n \mathfrak{u}_R(mn\overline{a}; q) \Big| \ll x(\log x)^{O(1)}(R^{-1} + M^{-1/2} + N^{-1/2}).$$

Proof. See [IK04, Theorem 17.4]. Only the case $r > R$ appears in our case. □

5.2. **First reductions.** First we apply two reductions, following Section V.2 of [Fou85] and Section 3 of [FI83]. We replace the sharp cutoff for the sum over q by a smooth function $\gamma(q)$; and we transfer the squareful part of n into the number a_2, allowing us to assume that n is squarefree. Note also that the statement of Theorem 5.1 is monotonically weaker as $\delta \to 0$, so that whenever needed, we will take the liberty of reducing the value of δ in a way that depends at most on η.

Proposition 5.3. *Let x, M, N, Q, R, η and the sequences (α_m) and (β_n) be as in Theorem 5.1. Assume that (β_n) is supported on squarefree integers. There exists $\delta > 0$ such that for any smooth function $\gamma : \mathbf{R}_+ \to [0, 1]$ with*

$$\mathbf{1}_{q\in(Q,2Q)} \leq \gamma(q) \leq \mathbf{1}_{q\in(Q/2,3Q/2]}, \tag{5.8}$$

and $\|\gamma^{(j)}\|_\infty \ll_j Q^{-j+B\delta j}$ for some $B \geq 0$ and all fixed $j \geq 0$, under the conditions (5.5)*, we have*

$$\sum_{\substack{q \\ (q,a_1a_2)=1}} \gamma(q) \sum_{\substack{n,m \\ (n,a_2)=1}} \alpha_m \beta_n \mathfrak{u}_R(mn\overline{a_1}a_2; q) \ll x(\log x)^{O(1)} R^{-1}. \tag{5.9}$$

The implicit constants depend on η, A (in (5.4)*), B and the function γ at most.*

Proof that Proposition 5.3 implies Theorem 5.1. We replace the sharp cutoff $Q < q \leq 2Q$ by a smooth weight $\gamma(q)$ such that

$$\mathbf{1}_{q\in(Q,2Q]} \leq \gamma(q) \leq \mathbf{1}_{q\in(Q(1-Q^{-10\delta}),2Q(1+Q^{-10\delta})]}.$$

We can pick γ such that $\|\gamma^{(j)}\|_\infty \ll_j Q^{-j+10\delta j}$ for all fixed $j \geq 0$. The error term in this procedure comes from the contribution of those integers q at the transition range $2Q < q \leq 2Q(1 + Q^{-10\delta})$ and $Q(1 - Q^{-10\delta}) \leq q \leq Q$. It is bounded by the triangle inequality, using our trivial bound (5.2) and following the reasonning of [BFI86, page 219 and 240], choosing $Q_0 = x^{10\delta}$ there. We obtain

$$\sum_{\substack{q \\ (q,a_1a_2)=1}} (\mathbf{1}_{Q<q\leq 2Q} - \gamma(q)) \sum_{\substack{n,m \\ (n,a_2)=1}} \alpha_m \beta_n \mathfrak{u}_R(mn\overline{a_1}a_2; q) \ll xR(\log x)^{O(1)} Q^{-10\delta}. \tag{5.10}$$

Given our hypotheses $R \leq x^\delta$ and $Q \geq x^{1/4}$, this is an acceptable error term.

Let $\mathcal{K}$ denote the set of squareful numbers:

$$\mathcal{K} = \{k \in \mathbf{N} : \ p|k \Rightarrow p^2|k\}.$$

Factor each integer n as $n = n'k$ with $\mu(n')^2 = 1$, $(n', k) = 1$ and $k \in \mathcal{K}$, so that $k \leq x^{1/3}$ and $(k, a_2) = 1$. Here μ is the Möbius function. There are only $O(K^{1/2})$ squareful numbers up to K [ES34], therefore

$$\sum_{\substack{k\geq K \\ k\in\mathcal{K}}} \frac{1}{k} \ll K^{-1/2} \qquad (K \geq 1).$$

Proceeding as in [Fou85, Section V.2] and using the trivial bound (5.2), we deduce for any $K\geq 1$,

$$\begin{aligned}&\sum_{\substack{q\\(q,a_1a_2)=1}}\gamma(q)\sum_{\substack{n,m\\(n,a_2)=1}}\alpha_m\beta_n\mathfrak{u}_R(mn\overline{a_1}a_2;q)\\ &=\sum_{\substack{k\leq K\\k\in\mathcal{K}\\(k,a_2)=1}}\sum_{\substack{q\\(q,a_1a_2)=1}}\gamma(q)\sum_{\substack{n,m\\(n,ka_2)=1}}\alpha_m\mu(n)^2\beta_{kn}\mathfrak{u}_R(mnk\overline{a_1}a_2;q)\\ &\qquad+O(Rx(\log x)^{O(1)}K^{-1/2}).\end{aligned}\tag{5.11}$$

We are left to analyze, for $k\in\mathcal{K}$, $k\leq K$, $(k,a_2)=1$, the sum

$$\sum_{\substack{q\\(q,a_1a_2)=1}}\gamma(q)\sum_{\substack{n,m\\(n,ka_2)=1}}\alpha_m\beta_{kn}\mu(n)^2\mathfrak{u}_R(mn\overline{a_1}ka_2;q).$$

Assume $K\leq x^{4\delta}$. For each fixed k, the sequences $(\alpha_m)_m$ and $(k^{-\delta}\mu(n)^2\beta_{kn})_n$ are supported in $m\in(M,2M]$ and $n\in(N/k,2N/k]$, respectively. We apply Proposition 5.3 with η replaced by $\eta/2$, N replaced by N/k and a_2 replaced by ka_2 (the factor $k^{-\delta}$ ensures that the condition (5.4) holds for $(k^{-\delta}\mu(n)^2\beta_{kn})_n$). If δ is small enough in terms of η, we obtain, uniformly for $k\leq K$,

$$\sum_{\substack{q\\(q,a_1a_2)=1}}\gamma(q)\sum_{\substack{n,m\\(n,ka_2)=1}}\alpha_m\beta_{kn}\mu(n)^2\mathfrak{u}_R(mn\overline{a_1}ka_2;q)\ll k^{-1+\delta}x(\log x)^{O(1)}R^{-1}.$$

Note that the sum $\sum_{k\in\mathcal{K}}k^{-1+\delta}$ converges. Inserting in (5.11), we obtain

$$\sum_{\substack{q\\(q,a_1a_2)=1}}\gamma(q)\sum_{\substack{n,m\\(n,a_2)=1}}\alpha_m\beta_n\mathfrak{u}_R(mn\overline{a_1}a_2;q)\ll x(\log x)^{O(1)}(R^{-1}+RK^{-1/2})$$

and so we conclude by the choice $K=R^4$. □

5.3. **Applying the dispersion method.** Let us prove Proposition 5.3. Recall that the sequence (β_n) is assumed to be supported on squarefree integers. Let $\mathcal{D}$ denote the left-hand side of (5.9). By the triangle inequality

$$|\mathcal{D}|=\Big|\sum_{(q,a_1a_2)=1}\gamma(q)\sum_{\substack{m,n\\(n,a_2)=1}}\alpha_m\beta_n\mathfrak{u}_R(mn\overline{a_1}a_2;q)\Big|\leq\sum_m\Big(|\alpha_m|\Big|\sum_q\sum_n\Big|\Big).$$

Let the function $\alpha(m)$ be $\mathcal{C}^\infty$ with $\alpha(m)\geq 1$ for $M<m\leq 2M$, supported inside $[M/2,2M]$ and such that $\|\alpha^{(j)}\|_\infty\ll_j M^{-j}$. Then by the Cauchy–Schwarz inequality and the hypothesis (5.4),

$$|\mathcal{D}|\ll(\log x)^{O(1)}M^{1/2}\Big(\mathcal{S}_1-2\,\mathfrak{Re}\,\mathcal{S}_2+\mathcal{S}_3\Big)^{1/2}\tag{5.12}$$

where

$$\mathcal{S}_1=\sum_{(q_1q_2,a_1a_2)=1}\gamma(q_1)\gamma(q_2)\sum_{\substack{n_1,n_2\\(n_1n_2,a_2)=1}}\beta_{n_1}\overline{\beta_{n_2}}\sum_{\substack{mn_1\equiv a_1\overline{a_2}\ (\mathrm{mod}\ q_1)\\mn_2\equiv a_1\overline{a_2}\ (\mathrm{mod}\ q_2)}}\alpha(m)$$

and $\mathcal{S}_2$ and $\mathcal{S}_3$ are defined similarly, replacing the sum over m by

$$\frac{1}{\varphi(q_2)}\sum_{\chi_2\in\mathcal{X}_{q_2}(R)}\chi(mn\overline{a_1}a_2)\sum_{n_1\equiv a_1\overline{a_2}\ (\mathrm{mod}\ q_1)}\alpha(m)\chi_2(m),$$

$$\frac{1}{\varphi(q_1)\varphi(q_2)}\sum_{\chi_1\in\mathcal{X}_{q_1}(R)}\sum_{\chi_2\in\mathcal{X}_{q_2}(R)}\chi_1(n_1\overline{a_1}a_2)\overline{\chi_2(n_2\overline{a_1}a_2)}\sum_{\substack{(mn_1,q_1)=1\\(mn_2,q_2)=1}}\alpha(m)\chi_1\overline{\chi_2}(m)$$

respectively. We will prove

$$\mathcal{S}_1-2\,\mathfrak{Re}\,\mathcal{S}_2+\mathcal{S}_3=O((\log x)^{O(1)}MN^2R^{-2}). \tag{5.13}$$

5.3.1. *Evaluation of* $\mathcal{S}_3$. The term $\mathcal{S}_3$ is defined by

$$\mathcal{S}_3=\sum_{(q_1q_2,a_1a_2)=1}\frac{\gamma(q_1)\gamma(q_2)}{\varphi(q_1)\varphi(q_2)}\sum_{\substack{\chi_1\in\mathcal{X}_{q_1}(R)\\\chi_2\in\mathcal{X}_{q_2}(R)}}\sum_{\substack{n_1,n_2\\(n_j,q_ja_2)=1}}\beta_{n_1}\overline{\beta_{n_2}}\sum_{(m,q_1q_2)=1}\alpha(m)\chi_1(mn_1\overline{a_1}a_2)\overline{\chi_2(mn_2\overline{a_1}a_2)}. \tag{5.14}$$

Let $W:=[q_1,q_2]$ and $H:=W^{1+\varepsilon}/M$. By Poisson summation (Lemma 3.1),

$$\begin{aligned}\sum_m\alpha(m)\chi_1\overline{\chi_2}(m)&=\frac{\widehat{\alpha}(0)}{W}\sum_{b\ (\mathrm{mod}\ W)^\times}\chi_1\overline{\chi_2}(b)\\&\quad+\frac{1}{W}\sum_{0<|h|\le H}\widehat{\alpha}\Big(\frac{h}{W}\Big)\sum_{b\ (\mathrm{mod}\ W)^\times}\mathrm{e}\Big(\frac{-bh}{W}\Big)\chi_1\overline{\chi_2}(b)+O_\varepsilon\Big(\frac{1}{W}\Big).\end{aligned}$$

The conductor of $\chi_1\overline{\chi_2}$ is at most R, so that [IK04, Lemma 3.2][8] yields

$$\sum_{b\ (\mathrm{mod}\ W)^\times}\mathrm{e}\Big(\frac{-bh}{W}\Big)\chi_1\overline{\chi_2}(b)\ll R^{1/2}\sum_{d|(h,W)}d.$$

We deduce

$$\sum_m\alpha(m)\chi_1\overline{\chi_2}(m)=\frac{\widehat{\alpha}(0)}{W}\sum_{b\ (\mathrm{mod}\ W)^\times}\chi_1\overline{\chi_2}(b)+O_\varepsilon(W^\varepsilon R^{1/2}).$$

The error term is $O(x^\delta)$ while the trivial bound is $M\ge x^{2/3}$. We deduce

$$\mathcal{S}_3=\widehat{\alpha}(0)X_3+O(MN^2x^{-1/2}),$$

where, having changed b to $ba_1\overline{a_2}$,

$$X_3:=\sum_{\substack{q_1,q_2\\(q_1q_2,a_1a_2)=1}}\frac{\gamma(q_1)\gamma(q_2)}{[q_1,q_2]\varphi(q_1)\varphi(q_2)}\sum_{\substack{\chi_1\in\mathcal{X}_{q_1}(R)\\\chi_2\in\mathcal{X}_{q_2}(R)}}\sum_{\substack{n_1,n_2\\(n_j,q_ja_2)=1}}\beta_{n_1}\overline{\beta_{n_2}}\sum_{b\ (\mathrm{mod}\ W)^\times}\chi_1(bn_1)\overline{\chi_2(bn_2)}.$$

By orthogonality,

$$\sum_{b\ (\mathrm{mod}\ W)^\times}\chi_1\overline{\chi_2}(b)=\varphi(W)\mathbf{1}_{\chi_1\sim\chi_2}$$

where by $\chi_1\sim\chi_2$ we mean that χ_1 and χ_2 are induced by the same primitive character – which necessarily has conductor dividing (q_1,q_2). Therefore,

$$\sum_{\substack{\chi_1\in\mathcal{X}_{q_1}(R)\\\chi_2\in\mathcal{X}_{q_2}(R)}}\chi_1(n_1)\overline{\chi_2(n_2)}\mathbf{1}_{\chi_1\sim\chi_2}=\sum_{\chi_0\in\mathcal{X}_{(q_1,q_2)}(R)}\chi_0(n_1\overline{n_2}).$$

Since $\varphi([q_1,q_2])=\varphi(q_1)\varphi(q_2)/\varphi((q_1,q_2))$, we deduce

$$X_3=\sum_{(q_1q_2,a_1a_2)=1}\frac{\gamma(q_1)\gamma(q_2)}{[q_1,q_2]\varphi((q_1,q_2))}\sum_{\chi_0\in\mathcal{X}_{(q_1,q_2)}(R)}\sum_{\substack{n_1,n_2\\(n_j,q_ja_2)=1}}\beta_{n_1}\overline{\beta_{n_2}}\chi_0(n_1\overline{n_2}). \tag{5.15}$$

[8]Note that in Lemma 3.2 of [IK04], $\tau(\chi)$ should read $\tau(\chi^*)$ and an additional factor $\chi^*(m/(dm^*))$ should appear in the summand.

5.3.2. *Evaluation of* $\mathcal{S}_2$. The term $\mathcal{S}_2$ is defined by
(5.16)
$$\mathcal{S}_2 = \sum_{(q_1q_2,a_1a_2)=1} \frac{\gamma(q_1)\gamma(q_2)}{\varphi(q_2)} \sum_{\substack{n_1,n_2\\(n_j,q_ja_2)=1}} \beta_{n_1}\overline{\beta_{n_2}} \sum_{\chi_2\in\mathcal{X}_{q_2}(R)} \sum_{m\equiv a_1\overline{a_2n_2} \pmod{q_1}} \alpha(m)\chi_2(mn_2\overline{a_1}a_2).$$
As before, let $W=[q_1,q_2]$ and $H=W^{1+\varepsilon}/M$. By Poisson summation,
$$\sum_{m\equiv a_1\overline{a_2n_1} \pmod{q_1}} \alpha(m)\chi_2(m) = \frac{\widehat{\alpha}(0)}{W} \sum_{\substack{b \pmod{W}^\times\\ b\equiv a_1\overline{a_2n_1} \pmod{q_1}}} \chi_2(b) + O_\varepsilon\Big(\mathcal{R}_2 + \frac{1}{W}\Big), \tag{5.17}$$
where
$$\mathcal{R}_2 := \frac{M}{W}\sum_{0<|h|\le H}\Big| \sum_{\substack{b \pmod{W}^\times\\ b\equiv a_1\overline{a_2n_1} \pmod{q_1}}} \chi_2(b)\mathrm{e}\Big(\frac{-bh}{W}\Big)\Big|. \tag{5.18}$$
We wish to express the sum over b as a complete sum over residues. We write $W=[q_1,q_2]=q_1'q_2'$, where $(q_2',q_1)=1$ and $q_1'|q_1^\infty$ (meaning that $p|q_1'\Rightarrow p|q_1$). Then $(q_1',q_2')=1$. Let
$$\psi : (\mathbf{Z}/q_1'\mathbf{Z})\times(\mathbf{Z}/q_2'\mathbf{Z}) \longrightarrow (\mathbf{Z}/W\mathbf{Z})$$
denote the canonical ring isomorphism (so ψ^{-1} is the projection map). Note that
$$b_2 \mapsto \chi_2(\psi(1,b_2))$$
defines a character $\pmod{q_2'}$ of conductor at most R. Finally, we have
$$\frac{1}{W} \equiv \frac{\overline{q_1'}}{q_2'} + \frac{\overline{q_2'}}{q_1'} \pmod{1}.$$
The sum over b in (5.18) is in absolute values at most
$$\sum_{\substack{b_1 \pmod{q_1'}^\times\\ b_1\equiv a_1\overline{a_2n_1} \pmod{q_1}}} \Big| \sum_{b_2 \pmod{q_2'}^\times} \chi_2(\psi(1,b_2))\mathrm{e}\Big(\frac{-b_2h\overline{q_1'}}{q_2'}\Big)\Big| \tag{5.19}$$
since $\psi(b_1,b_2)\equiv b_1 \pmod{q_1}$, and by factoring
$$\chi_2(\psi(b_1,b_2)) = \chi_2(\psi(b_1,1))\chi(\psi(1,b_2)).$$
The sum over b_2 in (5.19) is a Gauss sum; by [IK04, Lemma 3.2],
$$\Big| \sum_{b_2 \pmod{q_2'}^\times} \chi_2(\psi(1,b_2))\mathrm{e}\Big(\frac{-b_2h\overline{q_1'}}{q_2'}\Big)\Big| \le R^{1/2}\sum_{d|(h,q_2')} d. \tag{5.20}$$
Note that
$$\sum_{\substack{b_1 \pmod{q_1'}^\times\\ b_1\equiv a_1\overline{a_2n_1} \pmod{q_1}}} 1 = \frac{\varphi(q_1')}{\varphi(q_1)} = (q_2,q_1^\infty) \tag{5.21}$$
which is a shorthand for $\prod_{p^\nu\|q_2,\ p|q_1} p^\nu$. Multiplying (5.20) with (5.21) and summing over h, we obtain
$$\mathcal{R}_2 \ll_\varepsilon W^\varepsilon\tau(q_2)(q_2,q_1^\infty)R^{1/2}.$$
Inserting this estimate into (5.17) then (5.16), the error term contributes
$$\ll_\varepsilon R^{1/2}N^2W^\varepsilon \sum_{q_1,q_2\asymp Q}\frac{\tau(q_2)(q_2,q_1^\infty)}{q_2} \ll x^{\delta/2+\varepsilon}N^2Q.$$

In the last inequality we used standard facts about the kernel function $k(n) = \prod_{p|n} p$, for which we refer to [dB62]. The error term above is acceptable, since

$$x^{\delta/2}Q \leq x^{1/2+2\delta} \leq x^{2/3-2\delta} \leq MR^{-2}$$

if δ is small enough. We therefore have

$$\mathcal{S}_2 = \widehat{\alpha}(0)X_2 + O(MN^2R^{-2})$$

with (having changed b into $ba_1\overline{a_2n_2}$)

$$X_2 = \sum_{(q_1q_2,a_1a_2)=1} \frac{\gamma(q_1)\gamma(q_2)}{[q_1,q_2]\varphi(q_2)} \sum_{\substack{n_1,n_2\\(n_j,q_ja_2)=1}} \beta_{n_1}\overline{\beta_{n_2}} \sum_{\chi_2\in\mathcal{X}_{q_2}} \sum_{\substack{b \pmod{W}^\times\\ b\equiv\overline{n_1}n_2 \pmod{q_1}}} \chi_2(b).$$

Fix $\chi_2 \in \mathcal{X}_{q_2}$ and let $\widetilde{\chi}_2 \pmod{\widetilde{q}_2}$ be the primitive character inducing χ_2. Using orthogonality of characters $\pmod{(q_1,q_2)}$, the sum over b is

$$\sum_{\substack{b \pmod{W}^\times\\ b\equiv\overline{n_1}n_2 \pmod{q_1}}} \chi_2(b) = \frac{\varphi(q_2)}{\varphi((q_1,q_2))}\mathbf{1}_{\widetilde{q}_2|(q_0,q_1)}\widetilde{\chi}_2(\overline{n_1}n_2)$$

where we used the fact that $(n_1n_2,(q_1,q_2)) = 1$. Summing over $\chi_2 \in \mathcal{X}_{q_2}$, we obtain

$$\sum_{\chi_2\in\mathcal{X}_{q_2}} \sum_{\substack{b \pmod{W}^\times\\ b\equiv\overline{n_1}n_2 \pmod{q_1}}} \chi_2(b) = \frac{\varphi(q_2)}{\varphi((q_1,q_2))} \sum_{\chi_0\in\mathcal{X}_{(q_1,q_2)}} \chi_0(\overline{n_1}n_2),$$

and so $X_2 = X_3$.

5.4. **Second reduction.** We now wish to evaluate

$$\mathcal{S}_1 := \sum_{(q_1q_2,a_1a_2)=1} \gamma(q_1)\gamma(q_2) \sum_{\substack{n_1,n_2\\(n_j,q_1q_2a_2)=1\\ n_1\equiv n_2 \pmod{(q_1,q_2)}}} \beta_{n_1}\overline{\beta_{n_2}} \sum_{\substack{m\equiv a_1\overline{a_2n_1} \pmod{q_1}\\ m\equiv a_1\overline{a_2n_2} \pmod{q_2}}} \alpha(m).$$

The expected main term is $\widehat{\alpha(0)}X_1$, where

$$X_1 := \sum_{(q_1q_2,a_1a_2)=1} \frac{\gamma(q_1)\gamma(q_2)}{[q_1,q_2]} \sum_{\substack{n_1,n_2\\(n_j,q_j)=1\\ n_1\equiv n_2 \pmod{(q_1,q_2)}}} \beta_{n_1}\overline{\beta_{n_2}}. \tag{5.22}$$

For all integers q_0, n_0 with $(n_0,q_0) = 1$, let $\mathcal{S}_1(q_0,n_0)$ denote the contribution to $\mathcal{S}_1$ of those integers satisfying $(q_1,q_2) = q_0$ and $(n_1,n_2) = n_0$. Then we have

$$\begin{aligned}
|\mathcal{S}_1(q_0,n_0)| &\ll_\varepsilon x^\varepsilon \sum_{\substack{q_1,q_2\asymp Q/q_0\\(q_0q_2,a_2n_0)=1}} \sum_{\substack{n_1,n_2\asymp N/n_0\\ n_1\equiv n_2 \pmod{q_0}\\(n_2,q_0q_2)=1}} \sum_{\substack{a_2n_0n_2m\equiv a_1 \pmod{q_0q_2}\\ q_1|ma_2n_0n_1-a_1}} \alpha(m)\\
&\ll_\varepsilon x^\varepsilon \sum_{\substack{q_2\asymp Q/n_0\\(q_0q_2,a_2n_0)=1}} \sum_{\substack{n_1,n_2\asymp N/n_0\\ n_1\equiv n_2 \pmod{q_0}\\(n_2,q_0q_2)=1}} \sum_{ma_2n_0n_2\equiv a_1 \pmod{q_0q_2}} \alpha(m)\Big\{\mathbf{1}_{ma_2n_0n_1=a_1}\frac{Q}{q_0}\\
&\qquad\qquad + \mathbf{1}_{ma_2n_0n_1\neq a_1}\tau(|ma_2n_0n_1-a_1|)\Big\}\\
&\ll_\varepsilon x^\varepsilon\Big\{\frac{MN^2}{n_0^2q_0^2} + \frac{MN}{n_0q_0} + \frac{Q^2N}{n_0q_0}\Big\}.
\end{aligned}$$

Therefore, for some $\delta > 0$ and all $1 \leq K \leq x^\delta$, we have

$$\sum_{\substack{(q_0,n_0)=1\\ \max\{q_0,n_0\}>K}} |\mathcal{S}_1(q_0,n_0)| \ll_\varepsilon x^\varepsilon MN^2K^{-1}.$$

By choosing K appropriately, it will therefore suffice to show that

$$\mathcal{S}_1(q_0,n_0) = \widehat{\alpha}(0)X_1(q_0,n_0) + O(MN^2x^{-\delta}) \qquad (q_0,n_0 \leq x^\delta)$$

where $X_1(q_0,n_0)$ is the contribution to X_1 of indices with $(q_1,q_2)=q_0$ and $(n_1,n_2)=n_0$.

5.5. **Evaluation of $\mathcal{S}_1(q_0,n_0)$.** Let the integers q_0, n_0 be coprime, at most x^δ, such that $(q_0,a_1a_2)=(n_0,a_2)=1$. Let us rename q_1 into q_0q_1 and q_2 into q_0q_2, and similarly for n_1 and n_2. We wish to evaluate

$$\mathcal{S}_1(q_0,n_0) = \sum_{\substack{q_1,q_2\\ (q_1q_2,a_1a_2)=(q_1,q_2)=1}} \gamma(q_0q_1)\gamma(q_0q_2) \sum_{\substack{n_1,n_2\\ (n_0n_j,q_0q_ja_2)=1\\ (n_1,n_2)=1\\ n_1\equiv n_2 \pmod{q_0}}} \beta_{n_0n_1}\overline{\beta_{n_0n_2}} \sum_{m\equiv a_1\overline{a_2n_0n_j} \pmod{q_0q_j}} \alpha(m).$$

Using Poisson summation, we have

$$\mathcal{S}_1(q_0,n_0) = \widehat{\alpha(0)}X_1(q_0,n_0) + \mathcal{R}_1 + O_\varepsilon(x^\varepsilon\mathcal{R}_2)$$

where, having put $W=q_0q_1q_2$ and $H := W^{1+\varepsilon}M^{-1}$,

$$\mathcal{R}_1 = \sum_{q_1,q_2}\sum_{n_1,n_2} \gamma(q_0q_1)\gamma(q_0q_2)\beta_{n_0n_1}\overline{\beta_{n_0n_2}} \sum_{0<|h|\leq H} \frac{1}{W}\widehat{\alpha}\Big(\frac{h}{W}\Big)\mathrm{e}\Big(\frac{-h\mu}{W}\Big),$$

$$\mathcal{R}_2 = \sum_{q_1,q_2}\sum_{n_1,n_2}\frac{1}{W} \ll q_0^2N^2,$$

the summation conditions on q_j and n_j are the same as in the definition of $\mathcal{S}_1(q_0,n_0)$, and the residue class $\mu \pmod{W}$ satisfies

$$\mu \equiv a_1\overline{a_2n_0n_j} \pmod{q_0q_j} \qquad (j\in\{1,2\}).$$

We seek an error term $O(MN^2x^{-\delta})$. The contribution of $\mathcal{R}_2$ is acceptable.

We now focus on $\mathcal{R}_1$. Recall that β_n is non-zero only when n is squarefree (so that $(n_0,n_1)=1$). We have the equality modulo 1

$$\frac{\mu}{q_0q_1q_2} \equiv \frac{a_1}{q_0q_1q_2a_2n_0n_1} + a_1\frac{n_1-n_2}{q_0}\frac{\overline{q_1a_2n_0n_2}}{n_1q_2} - a_1\frac{\overline{q_0q_1q_2n_1}}{a_2n_0} \pmod 1.$$

Taking the exponential, we may approximate

$$\mathrm{e}\Big(\frac{a_1}{q_0q_1q_2a_2n_0n_1}\Big) = 1 + O\Big(\frac{|a_1|}{q_0q_1q_2a_2n_0n_1}\Big).$$

Inserting in $\mathcal{R}_1$, the error term contributes a quantity

$$\ll \frac{|a_1|q_0}{|a_2|n_0Q^2N}\frac{Q^2}{q_0^2}\frac{N^2}{n_0} \ll |a_1|N$$

which is clearly acceptable. We therefore evaluate

$$\mathcal{R}_1' := \sum_{q_1,q_2,n_1,n_2} \frac{\gamma(q_0q_1)\gamma(q_0q_1)}{q_0q_1q_2}\beta_{n_0n_1}\overline{\beta_{n_0n_2}}\widehat{\alpha}\Big(\frac{h}{q_0q_1q_2}\Big)\mathrm{e}\Big(-a_1h\frac{n_1-n_2}{q_0}\frac{\overline{q_1a_2n_0n_2}}{n_1q_2}+a_1h\frac{\overline{q_0q_1q_2n_1}}{a_2n_0}\Big).$$

Now we insert the definition of $\widehat{\alpha}$ as

$$\widehat{\alpha}\Big(\frac{h}{q_0q_1q_2}\Big) = q_0q_1q_2\int_{\mathbf{R}}\alpha(q_0q_1q_2\xi)\mathrm{e}(h\xi)\mathrm{d}\xi,$$

we detect the condition $(a_1, q_1q_2) = 1$ by Möbius inversion, and we split the sums over q_1, q_2 into congruence classes modulo n_0a_2. We obtain

$$|\mathcal{R}_1'| \ll (n_0|a_2|)^2\tau(|a_1|)^2\frac{Mq_0}{Q^2} \sup_{\xi\asymp Mq_0/Q^2} \sup_{\substack{\delta_1,\delta_2|a_1\\(\delta_1,\delta_2)=1\\(\delta_1\delta_2,n_0a_2)=1}} \sup_{\lambda_1,\lambda_2 \pmod{n_0a_2)^\times}} \mathcal{R}_1'' \tag{5.23}$$

where

$$\mathcal{R}_1'' := \sum_{\substack{q_1,q_2\\(\delta_1q_1,\delta_2q_2)=1\\q_j\equiv\lambda_j\overline{\delta_j} \pmod{n_0a_2}}} \gamma(q_0\delta_1q_1)\gamma(q_0\delta_2q_2) \sum_{\substack{n_1,n_2\\(n_0n_j,q_0\delta_jq_j)=1\\(n_1,n_2)=1\\n_1\equiv n_2 \pmod{q_0}}} \beta_{n_0n_1}\overline{\beta_{n_0n_2}}\times$$
$$\times\sum_{0<|h|\leq H}\alpha(\xi q_0\delta_1\delta_2q_1q_2)\mathrm{e}\Big(\xi h + a_1h\frac{\overline{q_0\lambda_1\lambda_2n_1}}{a_2n_0}\Big)\mathrm{e}\Big(-a_1h\frac{n_1-n_2}{q_0}\frac{\overline{a_2n_0n_2\delta_1q_1}}{n_1\delta_2q_2}\Big).$$

We write $\mathcal{R}_1''$ in the form (2.3), with

$$c\leftarrow q_2,\quad d\leftarrow q_1,\quad n\leftarrow -a_1h\frac{n_1-n_2}{q_0},\quad r\leftarrow a_2n_0n_2\delta_1,\quad s\leftarrow n_1\delta_2,\quad q\leftarrow n_0a_2, \tag{5.24}$$

taking the complex conjugate or not depending on the sign of $a_1h(n_1-n_2)$, and with the term

$$\gamma(q_0\delta_1q_1)\gamma(q_0\delta_2q_2)\alpha(\xi q_0\delta_1\delta_2q_1q_2)$$

playing the role of the function g. The derivative conditions (2.2) are satisfied with $\varepsilon_0 = B\delta$, by virtue of our hypothesis on γ. At this point, we are in a situation analogous to [BFI86, formula (13.2)]. Applying Theorem 2.1, and evaluating the terms as in [BFI86, page 241], we obtain

$$\mathcal{R}_1'' \ll x^{O(\delta)}\mathcal{A}^{1/2}\mathcal{B}^{1/2},$$

where $\mathcal{A}\ll HN^2$ is the contribution coming from $\|b_{N,R,S}\|_2^2$ in (2.3), and

$$\mathcal{B}\ll Q^2N^2N(H+N)+Q^3N^2\sqrt{H+N}+Q^2HN\ll (QN)^2\{N(H+N)+Q\sqrt{H+N}\}.$$

We have $H\ll x^{O(\delta)}N$, so that $\mathcal{B}\ll Q^2N^2x^{O(\delta)}(N^2+Q\sqrt{N})$ (compare with [BFI86, formula (13.4)]). Inserting in (5.23), we obtain

$$\mathcal{R}_1'\ll x^{O(\delta)}MN^2(Q^{-1}N^{3/2}+Q^{-1/2}N^{3/4})\ll x^{-\eta/2+O(\delta)}MN^2$$

by the hypothesis $N\leq Q^{2/3-\eta}$. Taking δ sufficiently small in terms of η, we have the required bound $O(MN^2x^{-\delta})$.

5.6. **The main terms.** The main terms X_1 and X_3 defined in (5.22) and (5.15) are real numbers. They combine to form

$$X_1 - X_3 = \sum_{(q_1q_2,a_1a_2)=1}\frac{\gamma(q_1)\gamma(q_2)}{[q_1,q_2]}\sum_{\substack{n_1,n_2\\(n_j,q_ja_2)=1}}\overline{\beta_{n_1}}\beta_{n_2}\mathfrak{u}_R(n_1\overline{n_2};(q_1,q_2)).$$

Notice the summands are zero unless $(q_1,q_2)>R$. We use Möbius inversion

$$\mathbf{1}_{(n_j,q_j)=1} = \sum_{d_j|(q_j,n_j)}\mu(d_j)$$

to detect the conditions $(n_j, q_j) = 1$, in order to separate the sums over n_1, n_2 from those over q_1, q_2. We insert the definition of $\mathfrak{u}_R$ in the form

$$\mathfrak{u}_R(n_1\overline{n_2}; q_0) = \frac{1}{\varphi((q_1, q_2))} \sum_{\substack{\chi \text{ primitive} \\ \mathrm{cond}(\chi) > R \\ \mathrm{cond}(\chi) | (q_1, q_2)}} \chi(\overline{n_1})\chi(n_2).$$

We can assume $(d_j, \mathrm{cond}(\chi)) = 1$ because of the factors $\chi(n_j)$. Quoting from [Ten95, Theorem I.5.4] the bound $\varphi(q) \gg q/\log\log q$, we obtain

$$X_1 - X_3 \ll (\log\log x) \sum_{R<r\leq Q} \sum_{\substack{d_1,d_2 \\ d_j \ll Q/r}} \Big(\sum_{\substack{q_1,q_2 \\ q_j \asymp Q \\ rd_j | q_j}} \frac{1}{q_1 q_2} \Big) \sum_{\substack{\chi \text{ primitive} \\ \chi \pmod r}} \prod_{j=1}^{2} \Big| \sum_{(n,a_2)=1} \beta_{d_j n} \chi(n) \Big|.$$

The sum over q_1, q_2 is $O(1/(r^2 d_1 d_2))$. By Cauchy–Schwarz, and the symmetry between n_1 and n_2, we obtain

$$X_1 - X_3 \ll (\log x)^2 \sum_{d \ll N} \frac{1}{d} \sum_{R<r\leq Q} \frac{1}{r^2} \sum_{\substack{\chi \text{ primitive} \\ \chi \pmod r}} \Big| \sum_{(n,a_2)=1} \beta_{dn}\chi(n) \Big|^2.$$

For all $t > R$, the multiplicative large sieve inequality (Lemma 3.3) and our hypothesis (5.4) yields

$$G(t) := \sum_{R<r\leq t} \sum_{\substack{\chi \text{ primitive} \\ \chi \pmod r}} \Big| \sum_{(n,a_2)=1} \beta_{dn}\chi(n) \Big|^2 \ll (\log x)^{O(1)} \tau(d)^{2A} (t^2 + N) N$$

after ignoring denominators d. We obtain by partial summation

$$X_1 - X_3 \ll (\log x)^2 \sum_{d \ll N} \frac{1}{d} \Big(\frac{G(Q)}{Q^2} + \int_R^Q \frac{G(t)}{t^3} \mathrm{d}t \Big) \ll (\log x)^{O(1)} (N + N^2 R^{-2}).$$

By hypothesis $R \leq x^{\delta}$, so we have the desired bound $X_1 - X_3 \ll N^2 R^{-2} (\log x)^{O(1)}$. Given $\widehat{\alpha}(0) \ll M$, our claimed estimate (5.13) is proved, and therefore Proposition 5.3 as well.

6. Application to the Titchmarsh divisor problem

The aim of this section is to justify Theorems 1.1 and 1.2. Recall the definition

$$T(x) := \sum_{1<n\leq x} \Lambda(n)\tau(n-1).$$

We let

$$\psi(x; q, a) := \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} \Lambda(n), \qquad \psi_q(x) := \sum_{\substack{n\leq x \\ (n,q)=1}} \Lambda(n), \qquad \psi(x,\chi) := \sum_{n\leq x} \Lambda(n)\chi(n).$$

Let us recall the following classical theorem of Page [IK04, Theorems 5.26, 5.28].

Lemma 6.1. *There is an absolute constant b such that for all $Q, T \geq 2$, the following holds. The function $s \mapsto \prod_{q\leq Q} \prod_{\chi \pmod q} L(s,\chi)$ has at most one zero $s = \beta$ satisfying $\mathfrak{Re}(s) > 1 - b/\log(QT)$ and $|\mathfrak{Im}(s)| \leq T$. If it exists, the zero β is real and it is the zero of a unique function $L(s, \widetilde{\chi})$ for some primitive real character $\widetilde{\chi}$.*

Given a large x, we shall say that $\widetilde{\chi}$ is x-exceptional if the above conditions are met with $Q = T = \mathrm{e}^{\sqrt{\log x}}$. For all $q \geq 1$ for which $\widetilde{q} | q$, we let $\widetilde{\chi}_q$ denote the character $\pmod q$ induced by $\widetilde{\chi}$.

6.1. **Primes in arithmetic progressions.** We deduce from the previous sections the following result about equidistribution of primes in arithmetic progressions.

Theorem 6.2. *Assume the GRH. For some $\delta > 0$, all $x \geq 1$, $Q \leq x^{1/2+\delta}$ and all integers $0 < |a_1|, |a_2| \leq x^\delta$,*

$$\sum_{\substack{q \leq Q \\ (q, a_1 a_2)=1}} \Big(\psi(x; q, a_1\overline{a_2}) - \frac{1}{\varphi(q)}\psi_q(x)\Big) \ll x^{1-\delta}.$$

Unconditionally, under the same assumptions,

$$\sum_{\substack{q \leq Q \\ (q, a_1 a_2)=1}} \Big(\psi(x; q, a_1\overline{a_2}) - \frac{\psi_q(x) + \mathbf{1}_{\widetilde{q}|q}\widetilde{\chi}(a_2\overline{a_1})\psi(x, \widetilde{\chi}_q)}{\varphi(q)}\Big) \ll x\mathrm{e}^{-\delta\sqrt{\log x}},$$

where the term $\psi(x; \widetilde{\chi}_q)$ is to be taken into account only if the x-exceptional character $\widetilde{\chi}$ exists.

Using the Dirichlet hyperbola method (see in particular section VII of [Fou85]), it follows that the same estimate holds on the condition $q \leq x^{1-\varepsilon}$ for any fixed $\varepsilon > 0$ (the implicit constants and δ may then depend on ε). Note however that the symmetry point is at $q \approx (x|a_2|)^{1/2}$, rather than $x^{1/2}$ (so the flexibility of taking Q somewhat larger than $x^{1/2}$ is not superfluous). We refer to [Fio12b] for more explanations on what happens when Q is very close to x.

As mentioned in the introduction, the uniformity in a_1 and a_2 is an interesting question. At the present state of knowledge, bounds coming from the theory of automorphic forms are typically badly behaved in that aspect. By using a more refined form of the combinatorial decomposition (6.4), Friedlander and Granville [FG92] prove that $|a_1| \leq x^{1/4-\varepsilon}$ is admissible for all $\varepsilon > 0$ (in the case $a_2 = 1$), with a somewhat larger error term.

For the application to the Titchmarsh divisor problem, the following slightly weaker statement suffices.

Proposition 6.3. *For some $\delta > 0$ and all $x \geq 2$, assuming the GRH, we have*

$$\sum_{\substack{q \leq \sqrt{x} \\ (q,a)=1}} \Big(\psi(x; q, a) - \psi(q^2; q, a) - \frac{\psi_q(x) - \psi_q(q^2)}{\varphi(q)}\Big) \ll x^{1-\delta}. \tag{6.1}$$

Unconditionally,

(6.2)

$$\sum_{\substack{q \leq \sqrt{x} \\ (q,a)=1}} \Big(\psi(x; q, a) - \psi(q^2; q, a) - \frac{\psi_q(x) - \psi_q(q^2)}{\varphi(q)} - \mathbf{1}_{\widetilde{q}|q}\overline{\chi(a)}\frac{\psi(x; \widetilde{\chi}_q) - \psi(q^2; \widetilde{\chi}_q)}{\varphi(q)}\Big) \ll x\mathrm{e}^{-\delta\sqrt{\log x}}.$$

We will focus here on proving Proposition 6.3 only, because the presentation is slightly simpler and addresses all the essential issues.

Proof of Proposition 6.3. Let $1 \leq R \leq x^{1/10}$ be a parameter. Let

$$\mathcal{S}_1 := \sum_{\substack{q \leq \sqrt{x} \\ (q,a)=1}} \sum_{\substack{q^2 < n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n).$$

By orthogonality of characters,

$$\mathcal{S}_1 = \sum_{\substack{q\leq\sqrt{x}\\(q,a)=1}} \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \sum_{q^2<n\leq x} \chi(n\overline{a})\Lambda(n) \tag{6.3}$$

We decompose $\mathcal{S}_1 = \mathcal{S}_1^- + \mathcal{S}_1^+$ where $\mathcal{S}_1^-$ is the contribution of those characters χ of conductor at most R, and

$$\mathcal{S}_1^+ = \sum_{\substack{q\leq\sqrt{x}\\(q,a)=1}} \sum_{q^2<n\leq x} \Lambda(n)\mathfrak{u}_R(n\overline{a};q).$$

We first focus of $\mathcal{S}_1^+$. By the Heath-Brown identity [BFI86, lemma 5] and a dichotomy argument similar to[FT85, Section 2.(a)], the problem is reduced to showing

$$\sum_{\substack{Q<q\leq 2Q\\(q,a)=1}} \sum\cdots\sum_{\substack{(1-\Delta)M_i<m_i\leq\min\{M_i,x^{1/4}\}\\(1-\Delta)N_i<n_i\leq N_i\\1\leq i\leq j}} \mu(m_1)\cdots\mu(m_j)(\log n_1)\mathfrak{u}_R(n_1m_1\cdots n_jm_j\overline{a};q) \ll x(\log x)^{O(1)}R^{-1} \tag{6.4}$$

where $j\in\{1,2,3,4\}$, $0<\Delta\leq 1/2$, and $Q, M_i, N_i \geq 1$ $(1\leq i\leq j)$ are real numbers such that

$$Q^2 \leq \prod_i M_iN_i \leq x, \qquad M_i \leq 2x^{1/4}.$$

Let $\eta>0$ be small. The contribution of tuples such that $\prod_i M_iN_i \leq x^{1-\eta}$ is trivially bounded by $O_\varepsilon(x^{1-\eta+\varepsilon})$ using Lemma 3.2. Suppose then $\prod_i M_iN_i > x^{1-\eta}$. For convenience we rename $x=\prod_i M_iN_i$. Our objective bound for (6.4) is $O(x^{1-\delta})$ and we now have $M_i \leq x^{1/4+\eta}$ if η is small enough.

Fix $\eta\in(0,1/100]$. At least one of the three following cases must hold:

(a) there exists an index k such that $N_k > x^{1-(2j-1)\eta}$,
(b) we have $\min\{N_k,N_{k'}\} > x^{1/3-\eta}$ for two indices $k\neq k'$,
(c) there exists an index k such that M_k or N_k lies in the interval $[x^\eta, x^{1/3-\eta}]$.

In case (a), our sum (6.4) is at most

$$\mathcal{S}_a := x^\varepsilon \sum_{\substack{Q<q\leq 2Q\\(q,a)=1}} \sum_{M/2<m\leq M} \Big| \sum_{(1-\Delta)N<n\leq N} \beta_n\mathfrak{u}_R(mn\overline{a};q)\Big| \tag{6.5}$$

with $\beta=\mathbf{1}$ or log, $MN=x$ and $N\geq x^{1-7\eta}$. Choose $\eta<1/30$, for the sum over n, we express $\mathfrak{u}_R$ as (5.3). Using

$$\sum_{\substack{n\leq z\\n\equiv a \pmod q}} 1 = \frac{z}{q} + O(1) \qquad (z\geq 1, (a,q)\in\mathbf{N}^2) \tag{6.6}$$

and partial summation in case $\beta=\log$, we get that the sum over n above is

$$\sum_{(1-\Delta)N<n\leq N} \beta_n\mathfrak{u}_R(mn\overline{a};q) \ll \log x + \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q\\1<\mathrm{cond}(\chi)\leq R}} \Big| \sum_{(1-\Delta)N<n\leq N} \beta_n\chi(n)\Big|.$$

For each χ in the above, the sum over n is estimated using Lemma 3.4 as

$$\sum_{(1-\Delta)N<n\leq N} \beta_n\chi(n) \ll R^{1/2}(\log x)^2\tau(q).$$

Dropping the condition $\mathrm{cond}(\chi) \leq R$, we obtain for (6.5) a crude bound

$$\mathcal{S}_a := x \ll_\varepsilon x^\varepsilon MQR^{1/2} \ll QR^{1/2}x^{8\eta} \ll x^{11/20+8\eta+\delta}$$

which is acceptable.

Consider case (b). Then the sum on the LHS of (6.4) is of the form

$$\mathcal{S}_b := \sum_{\substack{Q<q\leq 2Q\\(q,a)=1}} \sum\sum\sum_{\substack{(1-\Delta)N<n\leq N\\(1-\Delta)M<m\leq M\\(1-\Delta)^{2j-2}L<\ell\leq L}} \alpha(m)\beta(n)\gamma_\ell \mathfrak{u}_R(mn\ell\overline{a};q) \tag{6.7}$$

where $M, N > x^{1/3-\eta}$, $MNL = x$, α and β are either $\mathbf{1}$ or log, and γ_ℓ satisfies

$$|\gamma_\ell| \leq \tau_{2j-2}(\ell)\log \ell$$

By partial summation and upon rewriting the size restrictions on m, n, ℓ, q as differences of one-sided inequalities, it suffices to establish the bound

$$\mathcal{S}_b' := \sum_{\ell\leq L}\Bigg|\sum_{\substack{q\leq Q\\(q,a\ell)=1}}\sum_{m\leq M}\sum_{n\leq N}\mathfrak{u}_R(mn\ell\overline{a};q)\Bigg| \ll x^{1-\delta}$$

whenever $M, N > x^{1/3-2\eta}$ and $Q \leq 2\sqrt{x}$. Writing $\mathfrak{u}_R$ as in (5.3), we have by the triangle inequality

$$\mathcal{S}_b' \ll \mathcal{S}_{b1}' + \mathcal{S}_{b2}',$$

where

$$\mathcal{S}_{b1}' = \sum_{\ell\leq L}\Bigg|\sum_{\substack{q\leq Q\\(q,a\ell)=1}}\sum_{m\leq M}\sum_{n\leq N}\mathfrak{u}_1(mn\ell\overline{a};q)\Bigg|,$$

$$\mathcal{S}_{b2}' = \sum_{\ell\leq L}\sum_{q\leq Q}\frac{1}{\varphi(q)}\sum_{\substack{\chi \pmod q\\1<\mathrm{cond}(\chi)\leq R}}\Bigg|\sum_{m\leq M}\chi(m)\Bigg|\Bigg|\sum_{n\leq N}\chi(n)\Bigg|.$$

Theorem 7 of [BFI86] yields the acceptable bound $\mathcal{S}_{b1} \ll x^{1-\delta}$ as long as $\eta < 1/30$. In $\mathcal{S}_{b2}'$, by Lemma 3.4, the sums over m and n are majorized by $O(\tau(q)R^{1/2+\varepsilon})$. Dropping the condition $\mathrm{cond}(\chi) \leq R$, we obtain for (6.7) a bound

$$\mathcal{S}_{b2}' \ll_\varepsilon x^\varepsilon LRQ \ll_\varepsilon x^{11/12+5\eta}$$

which is also acceptable.

In case (c), we write our sum as

$$\sum_{\substack{Q<q\leq 2Q\\(q,a)=1}} \sum\sum_{\substack{(1-\Delta)^{2j-1}M<m\leq M\\(1-\Delta)N<n\leq N}} \alpha_m\beta_n\mathfrak{u}_R(mn\overline{a};q) \tag{6.8}$$

where $x^\eta \leq N \leq x^{1/3-\eta}$, so $M \geq x^{2/3}$. We may assume that $R \leq x^{\eta/2}$. If $Q \leq x^{1/2-\eta/2}$, then Lemma 5.2 is applicable. If on the contrary $x^{1/2-\eta/2} < Q \leq \sqrt{x}$, then Theorem 5.1 is applicable with $\eta \leftarrow \eta/2$ (assuming $|a| \leq x^{\delta/2}$ as we may). In both cases, we obtain that the quantity (6.5) is majorized by

$$\mathcal{S}_c \ll x(\log x)^{O(1)}R^{-1}.$$

Summarizing the above, we have obtained

$$\mathcal{S}_1^+ \ll x(\log x)^{O(1)}R^{-1}.$$

We consider now $\mathcal{S}_1^-$, which we recall is

$$\mathcal{S}_1^- = \sum_{\substack{q\leq\sqrt{x}\\(q,a)=1}} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q\\ \mathrm{cond}(\chi)\leq R}} \sum_{q^2<n\leq x} \Lambda(n)\chi(n\overline{a}).$$

First let us assume the GRH. Isolating the contribution of the principal character, we write

$$\mathcal{S}_1^- = \sum_{\substack{q\leq\sqrt{x}\\(q,a)=1}} \frac{\psi_q(x)-\psi_q(q^2)}{\varphi(q)} + \mathcal{S}_1^\flat,$$

say. For any non-trivial character $\chi \pmod q$ with $q \leq x$, the GRH [MV07, formula (13.19)] yields

$$\sum_{q^2<n\leq x} \chi(n)\Lambda(n) \ll x^{1/2}(\log x)^2.$$

We therefore have

$$\mathcal{S}_1^\flat \ll x^{1/2}(\log x)^2 \sum_{q\leq\sqrt{x}} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q\\ \mathrm{cond}(\chi)\leq R}} 1 \ll Rx^{1/2}(\log x)^3$$

which is acceptable. The choice $R = x^\delta$ for small enough δ concludes the proof of (6.1).

Unconditionally, for any $q \leq \mathrm{e}^{\sqrt{\log x}}$ and any non-principal, non x-exceptional character $\chi \pmod q$, we have by a straightforward adaptation of [MV07, Theorem 11.16] the estimate

$$\sum_{q^2<n\leq x} \chi(n)\Lambda(n) \ll x\mathrm{e}^{-c\sqrt{\log x}}$$

for some absolute constant $c > 0$. Choose $R = \mathrm{e}^{c\sqrt{\log x}/2}$. We write

$$\mathcal{S}_1^- = \sum_{\substack{q\leq\sqrt{x}\\(q,a)=1}} \frac{\psi_q(x)-\psi_q(q^2) + \mathbf{1}_{\widetilde{q}|q}\overline{\chi(a)}(\psi(x;\widetilde{\chi}_q)-\psi(q^2;\widetilde{\chi}_q))}{\varphi(q)} + \mathcal{S}_1^\flat + O(x\mathrm{e}^{-c\sqrt{\log x}/2}),$$

the error term being there to cover the trivial case when $\widetilde{q} > R$ (so $\widetilde{\chi}$ was not counted in $\mathcal{S}_1^-$). By the same computation as above,

$$\mathcal{S}_1^\flat \ll Rx(\log x)\mathrm{e}^{-c\sqrt{\log x}} \ll x\mathrm{e}^{-c\sqrt{\log x}/3}.$$

This concludes the proof of (6.2) hence of Proposition 6.3.

□

6.2. **Proof of Theorems 1.1 and 1.2.** It is now straightforward to deduce Theorems 1.1 and 1.2. By the Dirichlet hyperbola method [FT85, page 45], we have

$$T(x) = 2\sum_{q\leq\sqrt{x}} \Big(\psi(x;q,1)-\psi(q^2;q,1)\Big) + O(\sqrt{x}).$$

Assume first the GRH. Then Proposition 6.3 yields

$$T(x) = 2\sum_{q\leq\sqrt{x}} \frac{\psi_q(x)-\psi_q(q^2)}{\varphi(q)} + O(x^{1-\delta})$$

The GRH [MV07, formula (13.19)] allows us to deduce

$$T(x) = 2\sum_{q\leq\sqrt{x}} \frac{x-q^2}{\varphi(q)} + O(x^{1-\delta}).$$

The main term is computed using [Fou82, Lemme 6], which yields the claimed estimate.

Unconditionally, from Proposition 6.3, we merely have to add to our estimate for $T(x)$ the additional contribution of the x-exceptional character (if it exists), which takes the form

$$2\sum_{\substack{q\leq\sqrt{x}\\ \widetilde{q}\mid q}}\frac{\psi(x;\widetilde{\chi}_q)-\psi(q^2;\widetilde{\chi}_q)}{\varphi(q)} \tag{6.9}$$

We have from [MV07, Theorem 11.16]

$$\psi(x;\widetilde{\chi}_q)=-\frac{x^\beta}{\beta}+O(x\mathrm{e}^{-\delta\sqrt{\log x}})$$

and similarly

$$\psi(q^2;\widetilde{\chi}_q)=-\frac{q^{2\beta}}{\beta}+O(x\mathrm{e}^{-\delta\sqrt{\log x}})$$

at the possible cost of changing the numerical value of δ. We obtain that (6.9) equals

$$-\frac{2}{\beta}\sum_{\substack{q\leq\sqrt{x}\\ \widetilde{q}\mid q}}\frac{x^\beta-q^{2\beta}}{\varphi(q)}+O(x\mathrm{e}^{-\delta\sqrt{\log x}}).$$

The sums over q are computed using [Fou82, Lemme 6] (and partial summation in the form $x^\beta-q^{2\beta}=\beta\int_{q^2}^x t^{\beta-1}\mathrm{d}t$), which yields Theorem 1.2. Corollary 1.3 is straightforward.

There remains to justify Corollary 1.4. Note that $C_2(\widetilde{q})$ is absolutely bounded, while $\widetilde{q}\leq\mathrm{e}^{\sqrt{\log x}}$ by definition. Therefore $x^\beta\to\infty$, and $\beta\,\mathrm{li}(x^\beta)/x^\beta\sim(\log x)^{-1}$. We deduce

$$\frac{\log\widetilde{q}+C_2(\widetilde{q})-\gamma}{x^\beta/(\beta\,\mathrm{li}(x^\beta))}\underset{x\to\infty}{\longrightarrow}0$$

in an effective way. For x large enough, it is less than $1/3$ and Corollary 1.4 follows.

Remark. If we were to consider $\tau(n-a)$ instead of $\tau(n-1)$, for some a which is not a perfect square, then the Siegel zero contribution (if it existed) would have a twist by $\chi(a)$, which is *a priori* of unpredictable sign.

7. Application to correlation of divisor functions

In this section, we justify Theorem 1.5. The proof has the same structure as that of Theorems 1.1 and 1.2, replacing the function $\Lambda(n)$ by $\tau_k(n)$.

7.1. **An equidistribution estimate.** The analog of Theorem 6.2 is the following:

Theorem 7.1. *There exists $\eta>0$ such that under the conditions $k\geq 4$, $0<|a|\leq x^\eta$ and $Q\leq x^{1/2+\eta}$,*

$$\sum_{\substack{q\leq Q\\ (q,a)=1}}\Big(\sum_{\substack{n\leq x\\ n\equiv a \pmod q}}\tau_k(n)-\frac{1}{\varphi(q)}\sum_{\substack{n\leq x\\ (n,q)=1}}\tau_k(n)\Big)\ll x^{1-\eta/k}. \tag{7.1}$$

If the Lindelöf hypothesis is true for all Dirichlet L-functions, then the right-hand side can be replaced by $x^{1-\eta}$.

In order to simplify the presentation, we put

$$\mathcal{E} = \begin{cases} x & \text{if the generalized Lindelöf hypothesis is assumed,} \\ x^{1/k} & \text{unconditionally.} \end{cases}$$

To handle the small conductor case, we require the following.

Lemma 7.2. *For some $\delta > 0$ and any non-principal character $\chi \pmod q$ with $q \leq x$, of conductor $r \leq \mathcal{E}^\delta$ we have*

$$\sum_{n \leq x} \tau_k(n)\chi(n) \ll_k x\mathcal{E}^{-\delta}.$$

Proof. Starting from the representation

$$\sum_{n \leq x} \tau_k(n)\chi(n) = \frac{1}{2\pi i}\int_{1+1/(\log x)-i\infty}^{1+1/(\log x)+i\infty} L(s,\chi)^k \frac{x^s \mathrm{d}s}{s} \qquad (x \notin \mathbf{N}),$$

one may truncate the contour at $T = x^{\delta/k}$, and shift it to the abscissa $\mathfrak{Re}(s) = 1-\delta/k$. The convexity bound $|L(1-\delta/k+it,\chi)| \ll q^\varepsilon(r(|t|+1))^{c\delta/k+\varepsilon}$ (for some $c > 0$) yields the desired estimate if $\mathcal{E} = x^{1/k}$. If the Lindelöf hypothesis $L(\frac12+it,\chi) \ll (q(|t|+1))^\varepsilon$ is true, then one chooses $T = x^\delta$ and shifts the contour to $\mathfrak{Re}(s) = 1-\delta$, where the bound $L(1-\delta+it,\chi) \ll (q(|t|+1))^\varepsilon$ holds by convexity. □

7.1.1. *Small conductors.* Let $\mathcal{S}_0$ denote the quantity in the left-hand side of (7.1), and let $R \leq \mathcal{E}^\delta$. The contribution of those characters χ having conductors at most R is

$$\sum_{1<r\leq R}\sum_{\substack{\chi \pmod r \\ \chi \text{ primitive}}} \overline{\chi(a)} \sum_{\substack{q\leq Q \\ (q,a)=1 \\ r\mid q}} \frac{1}{\varphi(q)} \sum_{\substack{n\leq x \\ (n,q)=1}} \tau_k(n)\chi(n).$$

By Lemma 7.2 applied to the character $\pmod q$ induced by χ, we have a bound

$$x\mathcal{E}^{-\delta}\sum_{r\leq R}\sum_{\substack{\chi \pmod r \\ \chi\text{ primitive}}}\sum_{\substack{q\leq Q \\ r\mid q}} \frac{1}{\varphi(q)} \ll x\mathcal{E}^{-\delta}R(\log x)^2.$$

Letting $R = \mathcal{E}^{\delta/2}$, this is an acceptable error term. There remains to bound

$$\mathcal{S}_1 := \sum_{\substack{q\leq Q \\ (q,a)=1}}\sum_{n\leq x} \tau_k(n)\mathfrak{u}_R(n\overline{a};q).$$

7.1.2. *Dyadic decomposition.* We dyadically decompose in $\mathcal{S}_1$ the sums over q and n in (7.1), yielding an upper bound

$$\mathcal{S}_1 \ll (\log x)^2 \sup_{\substack{Q'\leq x^{1/2+\eta} \\ N\leq x}} \Big| \sum_{\substack{Q'<q\leq 2Q' \\ (q,a)=1}}\sum_{N<n\leq 2N} \tau_k(n)\mathfrak{u}_R(n\overline{a};q)\Big|. \tag{7.2}$$

Let $\eta > 0$ and assume throughout that δ is small with respect to η. When $N \leq x^{1-\eta}$, by the triangle inequality, our trivial bound (5.2) and Lemma 3.2, the sum over q and n above is $O_k(x^{1-\eta/2})$, so we may add the restriction $N > x^{1-\eta}$ in the supremum with an acceptable error. Then we relax the condition $Q' \leq x^{1/2+\eta}$ into $Q' \leq N^{1/2+2\eta}$. Renaming N into x, and expanding out $\tau_k(n)$, we obtain that it will suffice to prove

$$\mathcal{S}_2 := \sum_{\substack{Q<q\leq 2Q \\ (q,a)=1}}\sum_{x<n_1\cdots n_k\leq 2x} \mathfrak{u}_R(n_1\cdots n_k\overline{a};q) \ll x\mathcal{E}{-\eta} \tag{7.3}$$

under the constraints $|a| \leq x^{2\eta}$ and $Q \leq x^{1/2+2\eta}$. We decompose the sums over $n_1, \dots, n_k$ dyadically to obtain an upper bound

$$\mathcal{S}_2 \ll \mathcal{S}_3 := (\log x)^k \sup_{N_1,\dots,N_k \geq 1/2} \Big| \sum_{\substack{Q<q\leq 2Q \\ (q,a)=1}} \sum_{\substack{x<n_1\cdots n_k \leq 2x \\ N_j < n_j \leq 2N_j}} \mathfrak{u}_R(n_1 \cdots n_k \overline{a}; q) \Big|. \tag{7.4}$$

7.1.3. *Splitting cases.* Let the parameter $0 < \delta_1 < 1/100$ be fixed. We separate into two cases according to whether there is a subset $\mathcal{J} \subset \{1, \dots, k\}$ such that

$$\prod_{j \in \mathcal{J}} N_j \in (x^{\delta_1}, x^{1/3-\delta_1}],$$

or not. Suppose there is no such subset, and let

$$\mathcal{K} := \{j : \ 1 \leq j \leq k, \ N_j > x^{1/3-\delta_1}\}.$$

Necessarily card $\mathcal{K} \leq 3$. Since $N_j \leq x^{\delta_1}$ for each $j \notin \mathcal{K}$, and by assumption there is no subset $\mathcal{L} \subset \{1, \dots, k\} \smallsetminus \mathcal{K}$ such that $\prod_{j \in \mathcal{L}} N_j \in (x^{\delta_1}, x^{1/3-\delta_1}]$, it is necessarily the case that

$$\prod_{j \notin \mathcal{K}} N_j \leq x^{\delta_1}.$$

This implies card $\mathcal{K} \geq 1$. Define

$$\mathcal{W} := \{(u_n) \in \mathbf{C}^{\mathbf{N}} : \ |u_n| \leq 1 \quad (n \geq 1)\}.$$

Summarizing the above, we have

$$\mathcal{S}_3 \ll_{k,\varepsilon} x^{\varepsilon}(\mathcal{A} + \mathcal{B}_3 + \mathcal{B}_2 + \mathcal{B}_1), \tag{7.5}$$

where

$$\mathcal{A} = \sup_{\substack{x^{\delta_1} < N \leq x^{1/3-\delta_1} \\ MN=x \\ (\alpha_m),(\beta_n) \in \mathcal{W}}} \Big| \sum_{\substack{Q<q\leq 2Q \\ (q,a)=1}} \sum_{\substack{N<n\leq 2^k N \\ M2^{-k}<m<2M \\ x<mn\leq 2x}} \alpha_m \beta_n \mathfrak{u}_R(nm\overline{a}; q) \Big|,$$

$$\mathcal{B}_3 = \sup_{\substack{N_1,N_2,N_3 > x^{1/3-\delta_1} \\ MN_1N_2N_3=x \\ (\alpha_m) \in \mathcal{W}}} \Big| \sum_{\substack{Q<q\leq 2Q \\ (q,a)=1}} \sum_{\substack{N_j<n_j\leq 2N_j \\ M/8<m\leq 2M \\ x<mn_1n_2n_3<2x}} \alpha_m \mathfrak{u}_R(n_1n_2n_3m\overline{a}; q) \Big|,$$

$$\mathcal{B}_2 = \sup_{\substack{N_1,N_2 > x^{1/3-\delta_1} \\ N_1N_2 > x^{1-\delta_1} \\ MN_1N_2=x \\ (\alpha_m) \in \mathcal{W}}} \Big| \sum_{\substack{Q<q\leq 2Q \\ (q,a)=1}} \sum_{\substack{N_j<n_j\leq 2N_j \\ M/8<m\leq 2M \\ x<mn_1n_2<2x}} \alpha_m \mathfrak{u}_R(n_1n_2m\overline{a}; q) \Big|,$$

$$\mathcal{B}_1 = \sup_{\substack{N > x^{1-\delta_1} \\ MN=x \\ (\alpha_m) \in \mathcal{W}}} \Big| \sum_{\substack{Q<q\leq 2Q \\ (q,a)=1}} \sum_{\substack{N<n\leq 2 \\ M/8<m\leq 2M \\ x<mn<2x}} \alpha_m \mathfrak{u}_R(nm\overline{a}; q) \Big|.$$

We will focus on $\mathcal{A}$ and $\mathcal{B}_3$, since the treatment of $\mathcal{B}_1$ and $\mathcal{B}_2$ is analogous to $\mathcal{B}_3$ and actually simpler.

7.1.4. *Separation of variables.* Fix another small parameter $\delta_2 > 0$. We smoothen the cutoff using a smooth function $\phi : \mathbf{R} \to [0,1]$ with $\phi(\xi) = 1$ for $\xi \in [1,2]$, $\phi(\xi) = 0$ for $\xi \notin [1 - \mathcal{E}^{-\delta_2}, 2 + \mathcal{E}^{-\delta_2}]$, whose derivatives satisfy $\|\phi^{(j)}\|_\infty \ll_j \mathcal{E}^{j\delta_2}$. The cost of replacing in $\mathcal{A}$ and $\mathcal{B}_3$ the sharp cutoff condition $x < nm \leq 2x$ (resp. $x < n_1 n_2 n_3 m \leq 2x$) by $\phi(nm/x)$ (resp. $\phi(n_1 n_2 n_3 m/x)$) is at most $O(x\mathcal{E}^{-\delta_2/2})$, by trivially bounding the contribution of the transition ranges using Lemma 3.2.

Integration by parts shows that the Mellin transform $\breve{\phi}(s) = \int_0^\infty \phi(\xi)\xi^{s-1}\mathrm{d}\xi$ satisfies

$$\breve{\phi}(it) \ll \frac{\mathcal{E}^{5\delta_2}}{1 + |t|^5} \qquad (t \in \mathbf{R}).$$

We use the inversion formula $\phi(\xi) = (2\pi)^{-1} \int_{\mathbf{R}} \breve{\phi}(it)\xi^{-it}\mathrm{d}t$ at $\xi = nm/x$ (resp. $\xi = mn_1 n_2 n_3/x$) in the case of $\mathcal{A}$ (resp. $\mathcal{B}_3$), to obtain the upper bounds

$$\mathcal{A} \ll_k x\mathcal{E}^{-\delta_2/2} + \mathcal{E}^{5\delta_2} \sup_{\substack{x^{\delta_1} < N \leq x^{1/3-\delta_1}, \\ MN = x \\ (\alpha_m), (\beta_n) \in \mathcal{W}}} \Big| \sum_{\substack{Q < q \leq 2Q \\ (q,a)=1}} \sum_{\substack{N < n \leq 2^k N \\ M2^{-k} < m \leq 2M}} \alpha_m \beta_n \mathfrak{u}_R(mn\overline{a}; q) \Big|, \tag{7.6}$$

$$\begin{aligned} \mathcal{B}_3 \ll_k x\mathcal{E}^{-\delta_2/2} + \mathcal{E}^{5\delta_2} & \sup_{\substack{N_1, N_2, N_3 > x^{1/3-\eta}, \\ (\alpha_m) \in \mathcal{W},\ t \in \mathbf{R}}} \frac{1}{1 + |t|^3} \times \\ & \times \Big| \sum_{\substack{Q < q \leq 2Q \\ (q,a)=1}} \sum_{\substack{N_j < n_j \leq 2N_j \\ M/8 < m \leq 2M}} \alpha_m (n_1 n_2 n_3)^{it} \mathfrak{u}_R(n_1 n_2 n_3 m\overline{a}; q) \Big|. \end{aligned} \tag{7.7}$$

7.1.5. *The case of $\mathcal{A}$.* Let (α_m), (β_n) and N be given as in the supremum in (7.6). We wish to bound

$$\mathcal{S}_a := \sum_{\substack{Q < q \leq 2Q \\ (q,a)=1}} \sum_{\substack{N < n \leq 2^k N \\ M2^{-k} < m < 2M}} \alpha_m \beta_n \mathfrak{u}_R(mn\overline{a}). \tag{7.8}$$

By dyadic decomposition, enlarging our bound by a factor of k^2, we may assume the conditions are $N_1 < n \leq 2N_1$ and $M_1 < m \leq 2M_1$ for $M_1 N_1 \in [x2^{-k}, x2^{k+1}]$. Theorem 5.1 with $\eta \leftarrow \min\{\delta_1, 1/30\}$ gives the existence of $\delta_3 > 0$ depending on δ_1 such that (7.8) is majorized by $O(2^k x\mathcal{E}^{-\delta_3})$, on the condition that $|a| \leq 2^{-k}x^{\delta_3}$ and $Q \leq 2^{-k}x^{1/2+\delta_3}$, which are satisfied assuming $\eta < \delta_3/4$ and taking x large enough in terms of k.

If on the contrary $Q \leq x^{1/3}$, we appeal to Lemma 5.2 with $\eta \leftarrow \delta_1/k$ (or $\eta \leftarrow \delta_1$ if the Lindelöf hypothesis is assumed). We again obtain for (7.8) a bound

$$\mathcal{S}_a \ll_j 2^k x\mathcal{E}^{-\delta_3}$$

for some δ_3 (depending on δ_1).

Summarizing, we have obtained in any case

$$\mathcal{A} \ll_k x\mathcal{E}^{-\delta_2/2} + x\mathcal{E}^{5\delta_2 - \delta_3} \tag{7.9}$$

for $\delta_3 > 0$. Choosing δ_2 appropriately, it is an acceptable error term once we can prove that $\delta_1 > 0$ can be chosen independently of k.

7.1.6. *The case of $\mathcal{B}_3$.* Let (α_m), $N_1, N_2, N_3 > x^{1/3-\delta_1}$ and $t \in \mathbf{R}$ be as in supremum in (7.7). The quantity we wish to bound is at most

$$\mathcal{S}_b := \frac{1}{1+|t|^3} \sum_{M/8\le m\le 2M} \sum_{\substack{Q<q\le 2Q\\(q,am)=1}} \Big| \sum_{\substack{n_1,n_2,n_3\\N_j\le n_j\le 2N_j}} (n_1n_2n_3)^{it}\mathfrak{u}_R(n_1n_2n_3m\overline{a};q)\Big|$$

where $N_1N_2N_3M = x$ and $M < x^{3\delta_1}$. Writing $n_j^{it} = (2N_j)^{it} - it\int_{n_j}^{2N_j} z^{it-1}\mathrm{d}z$, the above is bounded by

$$\mathcal{S}_b \ll_\varepsilon \sup_{\substack{N_1',N_2',N_3'\\N_j<N_j'\le 2N_j}} \sum_{M/8\le m\le M} \sum_{\substack{Q<q\le 2Q\\(q,am)=1}} \Big| \sum_{\substack{n_1,n_2,n_3\\N_j\le n_j\le N_j'}} \mathfrak{u}_R(n_1n_2n_3m\overline{a})\Big| \tag{7.10}$$

Fix N_1', N_2', N_3' as in the supremum. Using (5.3) and the triangle inequality,

$$\mathcal{S}_b \le \mathcal{S}_b' + \mathcal{S}_b'',$$

where

$$\mathcal{S}_b' = \sum_{M/8\le m\le M} \sum_{\substack{Q<q\le 2Q\\(q,am)=1}} \Big| \sum_{\substack{n_1,n_2,n_3\\N_j\le n_j\le N_j'}} \mathfrak{u}_1(n_1n_2n_3m\overline{a})\Big|, \tag{7.11}$$

$$\mathcal{S}_b'' = \sum_{M/8<m\le M} \sum_{Q<q\le 2Q} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q\\1<\mathrm{cond}(\chi)\le R}} \prod_{j=1}^{3} \Big| \sum_{N_j<n\le N_j'} \chi(n)\Big|. \tag{7.12}$$

To $\mathcal{S}_b'$ we apply [BFI87, Lemma 2] for each q individually (note that this is a very deep result [FI85, HB86], relying on Deligne's proof of the Weil conjectures [Del74]). For some small, absolute δ_4, on the condition that $Q \le x^{1/2+\delta_4}$ (requiring $\eta < \delta_4/2$), the quantity (7.11) is bounded by

$$\mathcal{S}_b' \ll Mx^{1-\delta_4} \le x^{1-\delta_4+3\delta_1}. \tag{7.13}$$

Consider then $\mathcal{S}_b''$. By Lemma 3.4, each sum over n is bounded by $O_\varepsilon(x^\varepsilon R^{1/2})$, and so we obtain a bound

$$\mathcal{S}_b'' \ll_\varepsilon x^\varepsilon R^{5/2} M$$

which is absorbed in the term (7.13). Inserting in (7.7), we have obtained for $\mathcal{B}_3$ a bound

$$\mathcal{B}_3 \ll x\mathcal{E}^{-\delta_2} + \mathcal{E}^{5\delta_2}x^{1-\delta_4+3\delta_1}. \tag{7.14}$$

The terms $\mathcal{B}_2$ and $\mathcal{B}_1$ are shown in the same way to satisfy the same bound with $\delta_4 > 0$ absolute and small enough. Choosing our parameters adequately, we can choose absolute constants δ_1, δ_2, δ_3 in such a way that both bounds (7.14) and (7.9) are true and $O(x\mathcal{E}^{-\eta})$. Inserting back into (7.5) and (7.4), we obtain the claimed bound (7.3).

7.2. **Proof of Theorems 1.5 and 1.6.** As a last step, we deduce from Theorem 7.1 the estimate

$$\sum_{\substack{q\le\sqrt{x}\\(q,a)=1}} \Big(\sum_{\substack{n\le q^2\\n\equiv a \pmod q}} \tau_k(n) - \frac{1}{\varphi(q)} \sum_{\substack{n\le q^2\\(n,q)=1}} \tau_k(n)\Big) \ll_k x\mathcal{E}^{-\eta} \qquad (0<|a|\le x^\eta) \tag{7.15}$$

where as before $\mathcal{E} = x$ if the generalized Lindelöf is true and $\mathcal{E} = x^{1/k}$ otherwise. Let $\Delta \in (0, 1/10)$ be fixed and decompose the sums over q and n into intervals $((1+\Delta)^{-1}Q, Q]$ and $((1+\Delta)^{-1}N, N]$. Calling $\mathcal{S}_1'$ the left-hand side of (7.15), we have

$$\mathcal{S}_1' \ll \sum_{\substack{j_0, j_1 \geq 0 \\ Q=(1+\Delta)^{-j_0}\sqrt{x} \\ N=(1+\Delta)^{-j_1}x}} \Big| \sum_{(1+\Delta)^{-1}Q<q\leq Q} \sum_{\substack{(1+\Delta)^{-1}N<n\leq N \\ n\leq q^2}} \tau_k(n)\mathfrak{u}_1(n\overline{a}; q) \Big|,$$

where we used the notation (5.7). The inner sums are void if $Q^2 \leq N$ and the condition $n \leq q^2$ is automatically satisfied if $N \leq Q^2(1+\Delta)^{-2}$. The contribution of j_0, j_1 such that $(1+\Delta)^{-2}Q^2 \leq N \leq Q^2$ is at most

$$\sum_{\substack{q\leq\sqrt{x} \\ (q,a)=1}} \sum_{q^2(1+\Delta)^{-3}\leq n\leq q^2(1+\Delta)^2} \tau_k(n)|\mathfrak{u}_1(n\overline{a}; q)| \ll \Delta x(\log x)^k$$

by virtue of Lemma 3.2. Therefore

$$\mathcal{S}_1' \ll \Delta x(\log x)^k + (\log x)^2\Delta^{-2} \sup_{\substack{Q\leq\sqrt{x} \\ N\leq Q^2}} \Big| \sum_{(1+\Delta)^{-1}Q<q\leq Q} \sum_{(1+\Delta)^{-1}N<n\leq N} \tau_k(n)\mathfrak{u}_1(n\overline{a}; q) \Big|.$$

Let Q, N be as in the supremum, and let $\eta > 0$ be the real number given by Theorem 7.1. Lemma 3.2 gives the bound

$$\Big| \sum_{(1+\Delta)^{-1}Q<q\leq Q} \sum_{(1+\Delta)^{-1}N<n\leq N} \tau_k(n)\mathfrak{u}_1(n\overline{a}; q) \Big| \ll_\varepsilon x^\varepsilon N$$

which is acceptable if $N \leq x^{1-\eta/10}$. Suppose $N \geq x^{1-\eta/10}$, then Theorem 7.1 applies with $x \leftarrow N$ and yields a bound $O(x\mathcal{E}^{-\eta/10})$ for $|a| \leq x^{\eta/10}$. Therefore,

$$\mathcal{S}_1' \ll_{\varepsilon,k} x^{1+\varepsilon}\Delta + \Delta^{-2}x^{1+\varepsilon}\mathcal{E}^{1-\eta/10}.$$

Taking *e.g.* $\Delta = \mathcal{E}^{-\eta/30}$ and reinterpreting η, we have the claimed estimate (7.15).

From the Dirichlet hyperbola method, Theorem 7.1 and estimate (7.15), we deduce

$$\begin{aligned} \mathcal{T}_k(x) &= 2\sum_{q\leq\sqrt{x}} \sum_{\substack{q^2<n\leq x \\ n\equiv -1 \pmod q}} \tau_k(n) + O_\varepsilon(x^{1/2+\varepsilon}) \\ &= 2\sum_{q\leq\sqrt{x}} \frac{1}{\varphi(q)} \sum_{\substack{q^2<n\leq x \\ (n,q)=1}} \tau_k(n) + O(x\mathcal{E}^{-\delta}) \end{aligned}$$

The main terms are computed in [FT85, Théorème 2], with an error term $O(x^{1-\delta/k})$ (unconditionally). If one assumes the generalized Lindelöf hypothesis, then the proof is adapted in the following way. Under the hypotheses and in the notations of [FT85, Lemma 6], there holds $|\theta(p^\nu)| \leq Cp^{-\delta}\binom{k}{\lfloor k/2\rfloor}$ ([FT85, first display page 52]). Therefore the series $F_k(s)$ in [FT85, Lemma 7] is bounded in terms of k only in the half-place $\mathfrak{Re}(s) \geq 1-\delta/2$. In the proof of [FT85, Lemma 7], one chooses $T = x^{\delta/2}$ and shift the contour to $\mathfrak{Re}(s) = 1-\delta/2$, where the Lindelöf hypothesis implies $\zeta(s) \ll t^\varepsilon$ by convexity, to produce the conclusion

$$\sum_{n\leq x} \Psi(n)\tau_k(n) = xQ_{k-1}(\log x) + O_{\varepsilon,k}(x^{1-\delta/2+\varepsilon}).$$

The rest of the argument in Corollaries 1-2 of Lemma 7, and Corollary of Lemma 8 of [FT85] are transposed *verbatim* to yield

$$2\sum_{q\leq\sqrt{x}}\frac{1}{\varphi(q)}\sum_{\substack{q^2<n\leq x\\(n,q)=1}}\tau_k(n)=xP_k(\log x)+O_k(x^{1-c})$$

for some $c>0$, as claimed.

7.3. **Remark on the uniformity in a.** If we were to replace the shift $\tau(n+1)$ by $\tau(n+a)$, $0<|a|\leq x^\delta$, then the deduction of an asymptotic formula analogous to (1.4) from Theorem 7.1 goes along similar lines. We briefly indicate how one reduces to our previous setting. From Dirichlet's hyperbola method, the problem reduces to the evaluation of

$$\mathcal{S}_{k,a}(x)=2\sum_{q\leq\sqrt{x}}\sum_{\substack{q^2\leq n\leq x\\ n\equiv -a \pmod q}}\tau_k(n).$$

Extracting the largest factor $d_1|a^\infty$ from n, we rewrite this as

$$\mathcal{S}_{k,a}(x)=2\sum_{d_1|a^\infty}\tau_k(d_1)\sum_{q\leq\sqrt{x}}\sum_{\substack{q^2/d_1\leq n\leq x/d_1\\(n,a)=1\\ nd_1\equiv -a \pmod q}}\tau_k(n).$$

Writing $d_2:=(q,d_1)$, the congruence condition is equivalent to $d_2|a$ and

$$n\equiv -(a/d_2)\overline{(d_1/d_2)} \pmod{q/d_2}.$$

We therefore have

$$\mathcal{S}_{k,a}(x)=2\sum_{d_1|a^\infty}\tau_k(d_1)\sum_{d_2|(d_1,a)}\sum_{\substack{q\leq\sqrt{x}/d_2\\(q,d_1/d_2)=(q,a/d_2)=1}}\sum_{\substack{q^2/d_1\leq n\leq x/d_1\\(n,a)=1\\ n\equiv -(a/d_2)\overline{(d_1/d_2)} \pmod q}}\tau_k(n).$$

Summing for each d_j individually, the contribution of $d_1>x^\delta$ is bounded trivially using Lemma 3.2. When $d_1\leq x^\delta$, the sum over n and q is handled by an adequate generalization of Theorem 7.1, involving a congruence of the type $n\equiv b_1\overline{b_2} \pmod q$, as well as an additional coprimality condition $(n,b_3)=1$, for integers $|b_j|\leq x^\delta$. Our arguments readily adapt to account for both these modifications. Note however that it is now important that the method is able to handle values of the modulus q up to $x^{1/2+\delta}$, with δ independent of k (*cf.* the statement of Theorem 7.1).

References

[ABSR15] J. C. Andrade, L. Bary-Soroker, and Z. Rudnick, *Shifted convolution and the Titchmarsh divisor problem over $\mathbb{F}_q[t]$*, Philos. Trans. A Roy. Soc. London **373** (2015), no. 2040, to appear.

[BFI86] E. Bombieri, J. B. Friedlander, and H. Iwaniec, *Primes in arithmetic progressions to large moduli*, Acta Math. **156** (1986), no. 3-4, 203–251.

[BFI87] ———, *Primes in arithmetic progressions to large moduli. II*, Math. Ann. **277** (1987), no. 3, 361–393.

[BHM07a] V. Blomer, G. Harcos, and P. Michel, *Bounds for modular L-functions in the level aspect*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 5, 697–740.

[BHM07b] ———, *A Burgess-like subconvex bound for twisted L-functions*, Forum Math. **19** (2007), no. 1, 61–105, Appendix 2 by Z. Mao.

[BM15a] V. Blomer and D. Milićević, *Kloosterman sums in residue classes*, J. Eur. Math. Soc. **17** (2015), no. 1, 51–69.

[BM15b] ______, *The second moment of twisted modular L-functions*, Geom. Func. Anal. (2015), to appear.

[BV87] V. A. Bykovskiĭ and A. I. Vinogradov, *Inhomogeneous convolutions*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **160** (1987), no. Anal. Teor. Chisel i Teor. Funktsii. 8, 16–30, 296.

[CG01] J. B. Conrey and S. M. Gonek, *High moments of the Riemann zeta-function*, Duke Math. J. **107** (2001), no. 3, 577–604.

[dB62] N. G. de Bruijn, *On the number of integers $\leq x$ whose prime factors divide n.*, Illinois J. Math. **6** (1962), 137–141.

[Del74] P. Deligne, *La conjecture de Weil. I*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 43, 273–307.

[Des82] J.-M. Deshouillers, *Majorations en moyenne de sommes de Kloosterman*, Seminar on Number Theory, 1981/1982, Univ. Bordeaux I, Talence, 1982, pp. Exp. No. 3, 5.

[DFI02] W. Duke, J. B. Friedlander, and H. Iwaniec, *The subconvexity problem for Artin L-functions*, Invent. Math. **149** (2002), no. 3, 489–577.

[DI82a] J.-M. Deshouillers and H. Iwaniec, *An additive divisor problem*, J. London Math. Soc. (2) **26** (1982), no. 1, 1–14.

[DI82b] ______, *Kloosterman sums and fourier coefficients of cusp forms*, Invent. Math. **70** (1982), no. 2, 219–288.

[Dra15] S. Drappeau, *Théorèmes de type Fouvry–Iwaniec pour les entiers friables*, Compos. Math. (2015), to appear.

[ES34] P. Erdős and G. Szekeres, *Über die Anzahl der Abelschen Gruppen gegebener Ordnung und über ein verwandtes zahlentheoretisches Problem*, Acta Litt. Sci. Szeged **7** (1934), 95–102.

[Fel12] A. T. Felix, *Generalizing the Titchmarsh divisor problem*, Int. J. Number Theory **8** (2012), no. 3, 613–629.

[FG92] J. B. Friedlander and A. Granville, *Relevance of the residue class to the abundance of primes*, Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989), Univ. Salerno, Salerno, 1992, pp. 95–103.

[FI83] É. Fouvry and H. Iwaniec, *Primes in arithmetic progressions*, Acta Arith. **42** (1983), no. 2, 197–218.

[FI85] J. B. Friedlander and H. Iwaniec, *Incomplete Kloosterman sums and a divisor problem*, Ann. of Math. (2) **121** (1985), no. 2, 319–350, With an appendix by Bryan J. Birch and Enrico Bombieri.

[FI05] ______, *Summation formulae for coefficients of L-functions*, Canad. J. Math. **57** (2005), no. 3, 494–505.

[Fio12a] D. Fiorilli, *On a theorem of Bombieri, Friedlander, and Iwaniec*, Canad. J. Math. **64** (2012), no. 5, 1019–1035.

[Fio12b] ______, *Residue classes containing an unexpected number of primes*, Duke Math. J. **161** (2012), no. 15, 2923–2943.

[Fou82] É. Fouvry, *Répartition des suites dans les progressions arithmétiques*, Acta Arith. **41** (1982), no. 4, 359 382.

[Fou85] É. Fouvry, *Sur le problème des diviseurs de Titchmarsh*, J. Reine Angew. Math. **357** (1985), 51–76.

[FT85] É. Fouvry and G. Tenenbaum, *Sur la corrélation des fonctions de Piltz*, Rev. Mat. Iberoamericana **1** (1985), no. 3, 43–54.

[Hal67] H. Halberstam, *Footnote to the Titchmarsh-Linnik divisor problem*, Proc. Amer. Math. Soc. **18** (1967), 187–188.

[Har11] A. J. Harper, *On finding many solutions to S-unit equations by solving linear equations on average*, preprint (2011), arXiv, `http://arxiv.org/abs/1108.3819`.

[HB82] D. R. Heath-Brown, *Prime numbers in short intervals and a generalized Vaughan identity*, Canad. J. Math. **34** (1982), no. 6, 1365–1377.

[HB86] ______, *The divisor function $d_3(n)$ in arithmetic progressions*, Acta Arith. **47** (1986), no. 1, 29–56.

[IK04] H. Iwaniec and E. Kowalski, *Analytic number theory*, vol. 53, Cambridge Univ. Press, 2004.

[Iwa82] H. Iwaniec, *Mean values for Fourier coefficients of cusp forms and sums of Kloosterman sums*, Number theory days, 1980 (Exeter, 1980), London Math. Soc. Lecture Note Ser., vol. 56, Cambridge Univ. Press, Cambridge-New York, 1982, pp. 306–321.

[Iwa95] ———, *Introduction to the spectral theory of automorphic forms*, Biblioteca de la Revista Matemática Iberoamericana., Revista Matemática Iberoamericana, Madrid, 1995.

[Iwa97] ———, *Topics in classical automorphic forms*, Graduate Studies in Mathematics, vol. 17, American Mathematical Society, Providence, RI, 1997.

[Iwa02] ———, *Spectral methods of automorphic forms*, second ed., Graduate Studies in Mathematics, vol. 53, American Mathematical Society, Providence, RI; Revista Matemática Iberoamericana, Madrid, 2002.

[Kim03] H. H. Kim, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2, J. Amer. Math. Soc. **16** (2003), no. 1, 139–183, With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak.

[KL13] A. Knightly and C. Li, *Kuznetsov's trace formula and the Hecke eigenvalues of Maass forms*, Mem. Amer. Math. Soc. **224** (2013), no. 1055, vi+132.

[Lin63] Ju. V. Linnik, *The dispersion method in binary additive problems*, Translated by S. Schuur, American Mathematical Society, Providence, R.I., 1963.

[May15] J. Maynard, *Small gaps between primes*, Ann. of Math. (2) **181** (2015), no. 1, 383–413.

[Mot76] Y. Motohashi, *An induction principle for the generalization of Bombieri's prime number theorem*, Proc. Japan Acad. **52** (1976), no. 6, 273–275.

[MV07] H. L. Montgomery and R. C. Vaughan, *Multiplicative number theory. I. Classical theory*, Cambridge Studies in Advanced Mathematics, vol. 97, Cambridge University Press, Cambridge, 2007.

[PCF+14] D. H. J. Polymath, W. Castryck, É. Fouvry, G. Harcos, E. Kowalski, P. Michel, P. Nelson, E. Paldi, J. Pintz, A. V. Sutherland, T. Tao, and X.-F. Xie, *New equidistribution estimates of Zhang type*, Algebra Number Theory **8** (2014), no. 9, 2067–2199.

[Pit13] N. J. E. Pitt, *On an analogue of Titchmarsh's divisor problem for holomorphic cusp forms*, J. Amer. Math. Soc. **26** (2013), no. 3, 735–776.

[Pro03] N. V. Proskurin, *On general Kloosterman sums*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **302** (2003), no. Anal. Teor. Chisel i Teor. Funkts. 19, 107–134, 200.

[Rod65] G. Rodriquez, *Sul problema dei divisori di Titchmarsh*, Boll. Un. Mat. Ital. (3) **20** (1965), 358–366.

[Sar95] P. Sarnak, *Selberg's eigenvalue conjecture*, Notices Amer. Math. Soc. **42** (1995), no. 11, 1272–1277.

[Shi80] P. Shiu, *A Brun-Titchmarsh theorem for multiplicative functions*, J. Reine Angew. Math. **313** (1980), 161–170.

[Ten95] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Cambridge Studies in Advanced Mathematics, vol. 46, Cambridge University Press, Cambridge, 1995, Translated from the second French edition (1995) by C. B. Thomas.

[Tit30] E. C. Titchmarsh, *A divisor problem.*, Rend. Circ. Mat. Palermo **54** (1930), 414–429.

[Tit86] ———, *The theory of the Riemann zeta-function*, second ed., The Clarendon Press, Oxford University Press, New York, 1986, Edited and with a preface by D. R. Heath-Brown.

[Top15] B. Topacogullari, *The shifted convolution of divisor functions*, preprint (2015), arXiv, `http://arxiv.org/abs/1506.02608v1`.

[Wei48] A. Weil, *On some exponential sums*, Proc. Nat. Acad. Sci. U. S. A. **34** (1948), 204–207.

[Zha14] Y. Zhang, *Bounded gaps between primes*, Ann. of Math. (2) **179** (2014), no. 3, 1121–1174.

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